COS 488 - Homework 2 - Question 3

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5/5

Let $a_0 = a_1 = 0$ and $a_2 = 1$, and for all $n > 2$, let $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$. We can make this recurrence valid for all $n \geq 0$ by setting $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} + \delta_{n,2}$. Then, let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function for a_n, so that

$$A(z) = 3zA(z) - 3z^2A(z) + z^3A(z) + z^2.$$

Then,

$$A(z) = \frac{z^2}{1 - 3z + 3z^2 - z^3} = \frac{z^2}{(1-z)^3} = \sum_{n=2}^{\infty} \binom{n}{2} z^n,$$

so $a_n = \binom{n}{2}$ for all $n \geq 2$.

Now, let $a_0 = 0$ and $a_1 = a_2 = 1$, and for all $n > 2$, let $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$. We can make this recurrence valid for all $n \geq 0$ by setting $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} + \delta_{n,1} - 2\delta_{n,2}$. Then, let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function for a_n, so that

$$A(z) = 3zA(z) - 3z^2A(z) + z^3A(z) + z - 2z^2.$$

Then,

$$A(z) = \frac{z - 2z^2}{1 - 3z + 3z^2 - z^3} = \frac{z - 2z^2}{(1-z)^3} = \sum_{n=2}^{\infty} \binom{n}{2} z^{n-1} - 2\sum_{n=2}^{\infty} \binom{n}{2} z^n = z + \sum_{n=2}^{\infty} \left(\binom{n+1}{2} - 2\binom{n}{2} \right) z^n = z + \sum_{n=2}^{\infty} \frac{3n - n^2}{2} z^n,$$

so $a_n = \frac{3n-n^2}{2}$ for all $n \geq 2$.