5/5, slightly late submissions are fine

COS 488 - Homework 2 - Question 4

Matt Tyler

Hey grader. So here's the situation. As of the due date, I had the solution given on the second page of this document, and that is what I uploaded by the time it was due. A few minutes after the deadline, I realized I had been misreading the hint given for the problem, and I figured out how to do all of it. The full solution is given on the first page of this document. The lateness policy for this class is not mentioned anywhere on the syllabus, and the professors have not responded to my question on Piazza, so I'm hoping you could do me a huge favor and either grade the first page as a late solution or the second page as an on-time solution, depending on which one is less of a penalty. I totally understand if you don't want to take my word for it that I uploaded the second page on time, or if you don't want to do more work by reading two solutions, and I totally understand if you just mark me as being late and ignore the second page. Either way, thank you for your time.

By differentiating the equation

$$(1-z)^(-\alpha) = \sum_{n=0}^{\infty} \binom{-\alpha}{n}(-z)^n = \sum_{n=0}^{\infty} \frac{(-\alpha)(-\alpha-1)\dots(-\alpha-(n-1))}{n!}(-z)^n = \sum_{n=0}^{\infty} \frac{(\alpha)(\alpha+1)\dots(\alpha+(n-1))}{n!}z^n$$

with respect to α, we get

$$(1-z)^{-\alpha}\ln\frac{1}{1-z} = \sum_{n=0}^{\infty}\left(\frac{(\alpha)(\alpha+1)\dots(\alpha+(n-1))}{n!}\left(\frac{1}{\alpha}+\frac{1}{\alpha+1}+\dots+\frac{1}{\alpha+(n-1)}\right)\right)z^n,$$

so by letting $\alpha = \frac{1}{2}$, we have that

$$\begin{aligned}
[z^n]\frac{1}{\sqrt{1-z}}\ln\frac{1}{1-z} &= \frac{(1/2)(1/2+1)\dots(1/2+(n-1))}{n!}\left(\frac{1}{1/2}+\frac{1}{1/2+1}+\dots+\frac{1}{1/2+(n-1)}\right)\\
&= \frac{(2n-1)(2n-3)\dots(1)}{2^n n!}\left(\frac{2}{2n-1}+\frac{2}{2n-3}+\dots\frac{2}{1}\right)\\
&= \frac{(2n)!}{2^n n!(2n)(2n-2)\dots(2)}(2H_{2n}-H_n)\\
&= \frac{(2n)!}{2^{2n}n!n!}(2H_{2n}-H_n)\\
&= \binom{2n}{n}\frac{2H_{2n}-H_n}{4^n}
\end{aligned}$$

for all $n \geq 0$, where H_n is the n^{th} harmonic number.

COS 488 - Homework 2 - Question 4

Matt Tyler

By differentiating the equation

$$\frac{1}{2}(1-\sqrt{1-4z}) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} z^{n+1},$$

we get that

$$\frac{1}{\sqrt{1-4z}} = \sum_{n=0}^{\infty} \binom{2n}{n} z^n,$$

so

$$\frac{1}{\sqrt{1-z}} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{4^n}.$$

Therefore, we have that

$$\begin{aligned}
\frac{1}{\sqrt{1-z}} \ln \frac{1}{1-z} &= \left(\sum_{m=0}^{\infty} \binom{2m}{m} \frac{z^m}{4^m} \right) \left(\sum_{n=1}^{\infty} \frac{z^n}{n} \right) \\
&= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \binom{2m}{m} \frac{z^{m+n}}{4^m n} \\
&= \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \binom{2m}{m} \frac{z^n}{4^m (n-m)} \\
&= \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n-1} \binom{2m}{m} \frac{1}{4^m (n-m)} \right) z^n,
\end{aligned}$$

so for any $n > 0$,

$$[z^n] \frac{1}{\sqrt{1-z}} \ln \frac{1}{1-z} = \sum_{m=0}^{n-1} \binom{2m}{m} \frac{1}{4^m (n-m)},$$

while for $n = 0$,

$$[z^n] \frac{1}{\sqrt{1-z}} \ln \frac{1}{1-z} = 0.$$

Using mostly guessing and some computer aid, I was able to find the following closed-form expression for this summation, but I cannot prove that it is correct (H_k is the k^{th} Harmonic number):

$$\sum_{m=0}^{n-1} \binom{2m}{m} \frac{1}{4^m (n-m)} = \binom{2n}{n} \frac{1}{4^n} (2H_{2n} - H_n)$$

I have some partial progress towards a proof. Namely,

$$\begin{aligned}
\binom{2n}{n} \frac{1}{4^n} (2H_{2n} - H_n) &= \frac{(2n-1)!(H_{2n-1} - \frac{1}{2} H_{n-1})}{4^{n-1} n! (n-1)!} \\
&= \frac{(2n-1)!(H_{2n-1} - \frac{1}{2} H_{n-1})}{2^{n-1} n! (2n-2)(2n-4)\ldots(2)} \\
&= \frac{(2n-1)(2n-3)\ldots(1)\left(\frac{1}{2n-1} + \frac{1}{2n-3} + \cdots + \frac{1}{1}\right)}{2^{n-1} n!} \\
&= 2(-1)^n \binom{-1/2}{n} \left(\frac{1}{2n-1} + \frac{1}{2n-3} + \cdots + \frac{1}{1} \right) \\
&= 2\left([z^n] \frac{1}{\sqrt{1-z}} \right) \left(\frac{1}{2n-1} + \frac{1}{2n-3} + \cdots + \frac{1}{1} \right).
\end{aligned}$$