COS 488 Problem Set #2 Problem #3

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Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then by the recurrence relation in a_n,

$$f(z) - 3zf(z) + 3z^2 f(z) - z^3 f(z) = a_0 + (a_1 - 3a_0)z + (a_2 - 3a_1 + 3a_0)z^2$$
$$(1-z)^3 f(z) = z^2$$
$$f(z) = \frac{z^2}{(1-z)^3} = \sum_{n=2}^{\infty} \binom{n}{2} z^n$$

We can see this last identity by noting that

$$\begin{aligned}
\frac{z^{n-1}}{(1-z)^n} &= z\frac{1}{1-z}\frac{z^{n-2}}{(1-z)^{n-1}} \\
&= z\left(\sum_{j=0}^{\infty} z^j\right)\left(\sum_{j=n-2}^{\infty}\binom{j}{n-2}z^j\right) \\
&= z\sum_{j=n-2}^{\infty}\left(z^j\sum_{k=n-2}^{j}\binom{k}{n-2}\right) \\
&= \sum_{j=n-2}^{\infty}\binom{j+1}{n-1}z^{j+1} \\
&= \sum_{j=n-1}^{\infty}\binom{j}{n-1}z^j
\end{aligned}$$

Inductively, this shows $\frac{z^{n-1}}{(1-z)^n} = \sum_{j=n-1}^{\infty}\binom{j}{n-1}z^j$ for all $n \in \mathbb{N}$ (this trivially holds for $n = 1$).

As a result, $a_n = \binom{n}{2}$ for $n > 1$.

-2 Second part of the question?