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COS 488 Problem Set #2 Problem #4

Tim Ratigan

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First of all, note that if $f(z) = (1-z)^{-\alpha}$, then $f^{(n)}(z) = \alpha(\alpha+1)\dots(\alpha+n-1)(1-z)^{-(\alpha+n)}$ (this can be seen trivially by induction). As a result,

$$f(z) = (1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!} z^n$$

Furthermore,

$$\frac{\mathrm{d}((1-z)^{-\alpha})}{\mathrm{d}\alpha} = \frac{\mathrm{d}e^{-\alpha\log(1-z)}}{\mathrm{d}\alpha} = -\log(1-z)e^{-\alpha\log(1-z)} = (1-z)^{-\alpha}\log\left(\frac{1}{1-z}\right)$$

It follows that

$$(1-z)^{-\alpha}\log\left(\frac{1}{1-z}\right) = \sum_{n=1}^{\infty}\left(\left(\prod_{j=0}^{n-1}(\alpha+j)\right)\left(\sum_{j=0}^{n-1}\frac{1}{\alpha+j}\right)\frac{z^n}{n!}\right)$$

$$\frac{1}{\sqrt{1-z}}\log\left(\frac{1}{1-z}\right) = \sum_{n=1}^{\infty}\left(\left(\prod_{j=0}^{n-1}(j+1/2)\right)\left(\sum_{j=0}^{n-1}\frac{1}{j+1/2}\right)\frac{z^n}{n!}\right)$$

$$= z + z^2 + \frac{23}{24}z^3 + \frac{11}{12}x^4 + \dots$$