David Luo

Exercise 4.71

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First, we must rewrite the inner term into something we can work with, using Stirling's:

$$\frac{(N-k)^k(N-k)!}{N!} = exp(kln(N-k) + ln((N-k)!) - ln(N!))$$

$$= exp(kln(N-k) + (N-k+\tfrac{1}{2})ln(N-k) - N + k + ln(\sqrt{2\pi}) - (N+\tfrac{1}{2})lnN + N - ln(\sqrt{2\pi}) + O(\tfrac{1}{N}))$$

$$= exp(k + kln(N-k) + (N-k+\tfrac{1}{2})ln(N-k) - (N+\tfrac{1}{2})lnN + O(\tfrac{1}{N}))$$

$$= exp(k + (N+\tfrac{1}{2})ln(1-\tfrac{k}{N}) + O(\tfrac{1}{N}))$$

$$= exp(k - k - \tfrac{k^2}{2N} - \tfrac{k^3}{3N^2} - O(\tfrac{k^4}{N^3}) + O(\tfrac{1}{N}))$$

$$= e^{\frac{-k^2}{2N}}(1 - O(\tfrac{k^4}{N^3}) + O(\tfrac{1}{N}))$$

not sure why you dropped the k^3/N^2 term (it dominates the k^4/N^3 term), you should have O(k^3/N^2) instead of O(k^4/N^3). The bigger problem is that this only holds for k < k_0 for k_0 sufficiently small, but -0pt b/c I already took off for not talking about which k_0 in the Laplace method makes both sides work out

Then, we can use the Laplace method on the summation and our approximated inner term to show that the problem $P(N)$ is indeed equal to $\sqrt{\pi N/2}$:

$$P(N) \sim \sum_{k=0}^{N} e^{\frac{-k^2}{2N}} \sim \sqrt{N}\int_0^\infty e^{-x^2/2}dx = \sqrt{\pi N/2} + O(1)$$

-1pt for not explicitly mentioning the 1 to k_0 vs. k_0 to N split

-1pt for using ~ here--need to be more precise than that else you just get P(N) ~ sqrt(pi*N/2), which is weaker than P(N) = sqrt(pi*N/2) + O(1)