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Analytic Combinatorics Homework 3 Problem 4

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We have

$$\begin{aligned}\frac{(N-k)^k(N-k)!}{N!} &= \exp(k\ln(N-k)+\ln((N-k)!\,)-\ln(N!\,))\\ &= \exp(k\ln(N-k)+(N-k)\ln(N-k)-(N-k)+\ln\sqrt{2\pi(N-k)}\\ &\quad -N\ln(N)+N-\ln\sqrt{2\pi N}+O(1/N))\\ &= \exp\left(\left(N+\frac{1}{2}\right)\ln(N-k)-\left(N+\frac{1}{2}\right)\ln(N)+k+O\left(\frac{1}{N}\right)\right)\\ &= \exp\left(\left(N+\frac{1}{2}\right)\ln\left(1-\frac{k}{N}\right)+k+O\left(\frac{1}{N}\right)\right)\\ &= \exp\left(-k-\frac{k}{2N}-\frac{k^2}{N}+k+O\left(\frac{k^3}{N^2}\right)+O\left(\frac{1}{N}\right)\right)\\ &= \exp\left(-\frac{k}{2N}-\frac{k^2}{2N}+O\left(\frac{k^3}{N^2}\right)+O\left(\frac{1}{N}\right)\right)\\ &= e^{\frac{-k^2-k}{2N}}+O\left(\frac{k^3}{N^2}\right)+O\left(\frac{1}{N}\right).\end{aligned}$$

-0.5pt, this last equality only follows from the one before for k sufficiently small. You need to be more explicit about it here, even though you hint at it (but never explicitly say it)

Let us split the sum we wish to approximate into two sums, where $k_0 = N^{1/2}$.

$$\sum_{k=0}^{N-1}\frac{(N-k)^k(N-k)!}{N!}=\sum_{k=0}^{k_0-1}\frac{(N-k)^k(N-k)!}{N!}+\sum_{k=k_0}^{N-1}\frac{(N-k)^k(N-k)!}{N!}.$$

We now show that the tail (second sum) is $O(1)$. The terms are products of them form

$$\frac{N-k}{N-k+1}\cdot\frac{N-k}{N-k+2}\cdot\dots\cdot\frac{N-k}{N}.$$

This is clearly larger [smaller] for larger values of k, so the smallest [largest] term in the tail is when $k=k_0$, in which case we can bound the term (by ignoring the first half of the terms and noting that the last half are at most $\frac{N-k_0}{N-\frac{k_0}{2}}$) by

$$\left(\frac{N-k_0}{N-\frac{k_0}{2}}\right)^{k_0/2}=\left(1-\frac{1}{\frac{2N}{k_0}-1}\right)^{k_0/2}<\left(1-\frac{1}{\frac{2N}{k_0}}\right)^{k_0/2}=\left(1-\frac{1}{2k_0}\right)^{k_0/2}=O(1),$$

since this expression converges to $e^{1/4}$. Meanwhile the first sum is a sum of terms of the form $e^{\frac{-k^2-k}{2N}}+O\left(\frac{k^3}{N^2}\right)+O\left(\frac{1}{N}\right)$. Since $k<\sqrt{N}$, the second big-O gives us $\sqrt{N}$ terms of size at most proportional to $N^{-1/2}$, so the sum of that part is $O(1)$. The sum of the second big-O is $O(N^{-1/2})$ so we may ignore it. We are left with

$$\sum_{k=0}^{k_0-1}e^{\frac{-k^2-k}{2N}}.$$

-0.5pt, splitting doesn't work for k_0 = N^1/2, you flipped inequality sign (most people just cited results from book for what it's worth)

We may instead consider $e^{\frac{-k^2}{2N}}$, for changing the limits of the sum by 1 would produce a change in the value of $-k^2 - k$ of the same order as ignoring the linear term. Thus, this sum is, as in the Q-function, approximable by an integral, giving us

$$\sum_{k=0}^{k_0-1} e^{\frac{-k^2-k}{2N}} = O(1) + \sqrt{N}\int_0^\infty e^{\frac{-x^2}{2}}\,dx = \sqrt{\frac{\pi N}{2}} + O(1).$$

Combining this with the $O(1)$ we obtained from other parts of the summation still leaves us with $\sqrt{\frac{\pi N}{2}} + O(1)$, as desired.