COS 488
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PS3 - Q1

The ratio we are interested in is:

$$\frac{\alpha^N}{\beta^N} = \left(\frac{\alpha}{\beta}\right)^N$$

Note that using the exp-log trick, we get:

$$\left(\frac{\alpha}{\beta}\right)^N = e^{\ln\left(\frac{\alpha}{\beta}\right)^N} = e^{N(\ln(\alpha)-\ln(\beta))}$$

Denoting $\epsilon := \ln(\beta) - \ln(\alpha) > 0$ since we are given that $\alpha < \beta$, we can write:

$$\left(\frac{\alpha}{\beta}\right)^N = e^{-\epsilon N} = \left(e^{-N}\right)^\epsilon$$

Note that e^{-N} is exponentially small, i.e. $e^{-N} = O(N^{-M})$ for any $M > 0$, and so:

$$\left(e^{-N}\right)^\epsilon = \left(O(N^{-M})\right)^\epsilon = O(N^{-\epsilon M})$$

for any $M > 0$.

Since $\epsilon > 0$, we have that

$$\left(e^{-N}\right)^\epsilon = O(N^{M'})$$

for any $M' := -\epsilon M > 0$, and hence the ratio $\frac{\alpha^N}{\beta^N}$ is exponentially small, so α^N is exponentially small relative to β^N.

Next, we set $\beta = 1.2, \alpha = 1.1$.

When $N = 10$, the absolute error is given by:

$$\alpha^{10} + \beta^{10} - \beta^{10} = \alpha^{10} = 1.1^{10} \approx 2.59$$

and the relative error is then the absolute error divided by the exact value:

$$\frac{\alpha^{10}}{\alpha^{10} + \beta^{10}} = \frac{1.1^{10}}{1.1^{10} + 1.2^{10}} \approx 0.295$$

For $N = 100$, the absolute error is:

$$\alpha^{100} + \beta^{100} - \beta^{100} = \alpha^{100} = 1.1^{100} \approx 13781$$

while the relative error is:

$$\frac{\alpha^{100}}{\alpha^{100} + \beta^{100}} = \frac{1.1^{100}}{1.1^{100} + 1.2^{100}} \approx 0.000166$$