COS 488
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4/5

PS3 - Q4

For any term $\frac{(N-k)^k(N-k)!}{N!}$, we wish to obtain its asymptotic expansion. This can be done using the exp-log method:

$$\frac{(N-k)^k(N-k)!}{N!} = \exp(k\ln(N-k) + \ln((N-k)!) - \ln(N!)$$

Using Stirling's approximation, this gives

$$\exp(k\ln(N-k) + \ln((N-k)!) - \ln(N!)$$
$$= \exp(k\ln(N-k) + (N-k)\ln(N-k) - (N-k) + \ln(\sqrt{2\pi(N-k)}) + O(\frac{1}{N}) - (N\ln(N) - N + \ln(\sqrt{2\pi N}) + O(\frac{1}{N})))$$
$$= \exp(N\ln(N-k) + k + \ln\left(\frac{\sqrt{2\pi(N-k)}}{\sqrt{2\pi N}}\right) - N\ln N + O(\frac{1}{N}))$$
$$= \exp(N\ln\left(\frac{N-k}{N}\right) + k + \ln\left(\frac{\sqrt{N-k}}{\sqrt{N}}\right) + O(\frac{1}{N}))$$
$$= \exp((N+0.5)\ln\left(1-\frac{k}{N}\right) + k + O(\frac{1}{N}))$$
$$= \exp((N+0.5)(-\frac{k}{N} - \frac{k^2}{2N^2} + O(\frac{k^3}{N^3})) + k + O(\frac{1}{N}))$$
$$= \exp(-k - \frac{k^2}{2N} + O(\frac{k^3}{N^2})) + k + O(\frac{k}{N}))$$
$$= \exp(-\frac{k^2}{2N} + O(\frac{k^3}{N^2})) + O(\frac{k}{N}))$$
$$= e^{-k^2/2N}(1 + O(\frac{k^3}{N^2}) + O(\frac{k}{N}))$$

-1pt, last equality is only true for sufficiently small k, if it were just about taking k large enough we'd take k_0 = N--we also need k_0 << N^2/3

Hence, we can write that

$$\frac{(N-k)^k(N-k)!}{N!} = e^{-k^2/2N}(1 + O(\frac{k^3}{N^2}) + O(\frac{k}{N}))$$

Notice that as k gets close to N, this expression becomes 0. Now, we can approximate the sum using the Laplace method. First we pick a k_0 large enough so that we restrict the range to the largest summands and the tail is exponentially small, so:

$$P(N) = \sum_{0\le k<N} \frac{(N-k)^k (N-k)!}{N!} = \sum_{0\le k\le k_0} \frac{(N-k)^k (N-k)!}{N!} + \sum_{k_0<k<N} \frac{(N-k)^k (N-k)!}{N!}$$

Then we plug in our asymptotic expansion:

$$\sum_{0\le k\le k_0} \frac{(N-k)^k (N-k)!}{N!} = \sum_{0\le k\le k_0} e^{-k^2/2N}(1 + O(\frac{k^3}{N^2}) + O(\frac{k}{N}))$$

Now, the tail of this sum is also be exponentially small, so we can approximate

$$P(N) = \sum_{k\ge 0} e^{-k^2/2N}(1 + O(\frac{k^3}{N^2}) + O(\frac{k}{N}))$$

Which can then be approximated via an integral:

$$P(N) = \int_0^\infty e^{-k^2/2N}(1 + O(\frac{k^3}{N^2}) + O(\frac{k}{N}))dk$$

$$= \sqrt{N}\int_0^\infty e^{-x^2/2}(1 + O(\frac{x^3}{\sqrt{N}}) + O(\frac{x}{\sqrt{N}}))dx$$

$$= \sqrt{\pi N/2} + \int_0^\infty (O(x^3) + O(x))dx$$

where we have changed variables to $x = k/\sqrt{N}$. Now note that the integral term does not depend on either k or N, and is hence a constant, O(1), so

$$P(N) = \sqrt{\pi N/2} + O(1)$$