COS 488
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PS3 - Q2

Notice that we can write

$$\frac{N}{N-1} = \frac{1}{1-\frac{1}{N}} = 1 + \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + O(\frac{1}{N^4})$$

And we can write

$$\ln\frac{N}{N-1} = \ln(1 + (\frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + O(\frac{1}{N^4})))$$

If we treat the sum of the terms following 1 as x which is close to 0, we can write

$$\ln(1 + (\frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + O(\frac{1}{N^4})))$$

$$= \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + O(\frac{1}{N^4}) - \frac{1}{2}(\frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + O(\frac{1}{N^4}))^2 + \frac{1}{3}(\frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + O(\frac{1}{N^4}))^3 + O(\frac{1}{N^4})$$

$$= \frac{1}{N} + \frac{1}{N^2} - \frac{1}{2}\frac{1}{N^2} + \frac{1}{N^3} - \frac{1}{2}\frac{2}{N^3} + \frac{1}{3}\frac{1}{N^3} + O(\frac{1}{N^4})$$

$$= \frac{1}{N} + \frac{1}{2N^2} + \frac{1}{3N^3} + O(\frac{1}{N^4})$$

Alternatively, we can also see this expansion by writing:

$$\ln\frac{N}{N-1} = \ln\frac{1}{1-\frac{1}{N}} = \ln 1 - \ln(1 - \frac{1}{N}) = -(-\frac{1}{N} - \frac{1}{2N^2} - \frac{1}{3N^3} + O(\frac{1}{N^4})$$

$$= \frac{1}{N} + \frac{1}{2N^2} + \frac{1}{3N^3} + O(\frac{1}{N^4})$$

And hence, our asymptotic expansion for $\frac{N}{N-1}\ln\frac{N}{N-1}$ is:

$$\frac{N}{N-1}\ln\frac{N}{N-1} = (1 + \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + O(\frac{1}{N^4})) * (\frac{1}{N} + \frac{1}{2N^2} + \frac{1}{3N^3} + O(\frac{1}{N^4}))$$

$$= \frac{1}{N} + \frac{1}{2N^2} + \frac{1}{N^2} + \frac{1}{3N^2} + \frac{1}{2N^3} + \frac{1}{N^3} + O(\frac{1}{N^4})$$

$$= \frac{1}{N} + \frac{3}{2N^2} + \frac{11}{6N^3} + O(\frac{1}{N^4})$$