Homework 3: Exercise 4.9

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If $\alpha < \beta$, we will show that $\frac{\alpha^N}{\beta^N}$ is exponentially small.

Suppose, on the contrary, that there exists $M > 0$ such that $\frac{\alpha^N}{\beta^N} \geqslant N^{-M}$.

Taking log base β/α of both sides, we have

$$\begin{aligned}\left(\frac{\alpha}{\beta}\right)^N &\geqslant N^{-M} \\ \left(\frac{\beta}{\alpha}\right)^N &\leqslant N^M \\ N &\leqslant M\log(N) \\ \frac{N}{\log N} &\leqslant M.\end{aligned}$$

However, since the left-hand-side goes to infinity in the limit, so there exists some N_0 such that for $N > N_0$, $\frac{N}{\log N} \geqslant M$, which is a contradiction. Therefore, there is no fixed M that makes $\frac{\alpha^N}{\beta^N} \geqslant N^{-M}$ hold.

When $\alpha = 1.1$ and $\beta = 1.2$, approximating $\alpha^N + \beta^N$ as β^N gives the following error terms:

- When $N = 10$:
  - Absolute error: $\alpha^N + \beta^N - \beta^N = \alpha^N = 1.1^{10} = 2.59374$.
  - Exact value: $\alpha^N + \beta^N = 1.1^{10} + 1.2^{10} = 8.78548$.
  - Relative error: $\frac{2.59374}{8.78548} = 0.29523$.
- When $N = 100$:
  - Absolute error: $\alpha^N + \beta^N - \beta^N = \alpha^N = 1.1^{100} = 13780.61234$.
  - Exact value: $\alpha^N + \beta^N = 1.1^{100} + 1.2^{100} = 82831755.13435$.
  - Relative error: $\frac{13780.61234}{82831755.13435} = 0.0001663687$.

The error seems negligible for large N, as expected.