Homework 3: Exercise 4.71

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First, we will find a simplified form for each term of the sum:

$$
\begin{aligned}
\frac{(N-k)^k(N-k)!}{N!} &= \exp\left(k\ln(N-k)+\ln(N-k)!-\ln N!\right) && \text{[exp-log]}\\
&= \exp\Bigg(k\ln(N-k)+(N+\frac{1}{2}-k)\ln(N-k)-(N-k)\\
&\quad -(N+\frac{1}{2})\ln N+N+O(\frac{1}{N})\Bigg) && \text{[Stirling]}\\
&= \exp\left((N+\frac{1}{2})\ln(N-k)+k-(N+\frac{1}{2})\ln N+O(\frac{1}{N})\right) && \text{[algebra]}\\
&= \exp\left((N+\frac{1}{2})\ln(\frac{N-k}{N})+k+O(\frac{1}{N})\right) && \text{[algebra]}\\
&= \exp\left((N+\frac{1}{2})\ln(1-\frac{k}{N})+k+O(\frac{1}{N})\right) && \text{[algebra]}
\end{aligned}
$$

We can expand $\ln(1+\frac{k}{N}) = -\frac{k}{N}-\frac{k^2}{2N^2}+O(\frac{k^3}{N^3})$, so

$$
\begin{aligned}
\frac{(N-k)^k(N-k)!}{N!} &= \exp\left((N+\frac{1}{2})\ln(1-\frac{k}{N})+k+O(\frac{1}{N})\right)\\
&= \exp\left((N+\frac{1}{2})\left(-\frac{k}{N}-\frac{k^2}{2N^2}+O(\frac{k^3}{N^3})\right)+k+O(\frac{1}{N})\right) && \text{[expand]}\\
&= \exp\left(-k-\frac{k^2}{2N}+O(\frac{k^3}{N^2})-\frac{k}{2N}-\frac{k^2}{4N^2}+O(\frac{k^3}{N^3})+k+O(\frac{1}{N})\right) && \text{[multiply]}\\
&= \exp\left(\frac{-k^2}{2N}+O(\frac{k^3}{N^2})+O(\frac{k}{N})\right) && \text{[simplify]}\\
&= e^{\frac{-k^2}{2N}}+O(\frac{k^3}{N^2})+O(\frac{k}{N})
\end{aligned}
$$

-0.5 pt, step only follows for sufficiently small k

Note that this is asymptotically identical to the Ramanujan Q-distribution, as outlined in Slide 33 of lecture 4. Therefore, summing up the function over all k, the same analysis outlined in Slide 38 of Lecture 4 goes through, giving

$$P(N) \equiv \sum_{0\leqslant k<N} \frac{(N-k)^k(N-k)!}{N!} \sim Q(N) \sim \sqrt{\pi N/2}.$$

-1pt, slides give P(N) ~ sqrt(pi*N/2), not P(N) = sqrt(pi*N/2) + O(1) as desired

More specifically, since we can tolerate constant error, changing the range of the sum from $0 \leqslant k < N$ to $1 \leqslant k \leqslant N$ does not change the asymptotics. Similar to lecture, we can use Laplace method and split up the sum into two parts, for $k < k_0 = o(n^{2/3})$ and the tail, which is exponentially small and thus negligible. Euler-Maclaurin summation theorem then allows us to express the sum as an integral of points with step $1/\sqrt{N}$, so

$$P(N) \sim \sqrt{N} \int_0^\infty e^{\frac{-x^2}{2}} dx = \sqrt{\pi N/2}.$$