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COS 488/MAT 474
Problem Set 3, Q1

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**AofA Exercise 4.9** If $\alpha < \beta$, show that $\alpha^n$ is exponentially small relative to $\beta^N$. For $\beta = 1.2$ and $\alpha = 1.1$, find the absolute and the relative errors when $\alpha^N + \beta^N$ is approximated by $\beta^N$, for $N = 10$ and $N = 100$.

*Solution.* Suppose $\alpha < \beta$. We need to show that $\frac{\alpha^N}{\beta^N} = O(N^{-M})$ for <u>all</u> $M > 0$. We take as known the result that $e^{-N}$ is exponentially small. Let $c = \ln \frac{\beta}{\alpha}$, and note that $c > 0$ since $\frac{\beta}{\alpha} > 1$. Then for all $M > 0$, we have:

$$\frac{\alpha^N}{\beta^N} = \left(\frac{\alpha}{\beta}\right)^N = e^{N \ln \frac{\alpha}{\beta}} = e^{-cN} = (e^{-N})^c = \left(O(N^{-\frac{M}{c}})\right)^c = O(N^{-M}).$$

Therefore $\alpha^N$ is exponentially small relative to $\beta^N$.

As an example, take $\alpha = 1.1$ and $\beta = 1.2$.

When $N = 10$, we have $\alpha^N + \beta^N \approx 8.785$ and $\beta^N \approx 6.192$. The absolute error is $|\alpha^N| \approx 2.594$, and the relative error is $\left|\frac{\alpha^N}{\alpha^N+\beta^N}\right| \approx 0.295$.

When $N = 100$, we have $\alpha^N + \beta^N \approx 8.28318 \times 10^7$ and $\beta^N \approx 8.28180 \times 10^7$. The absolute error is $|\alpha^N| \approx 0.00138 \times 10^7$, and the relative error is $\left|\frac{\alpha^N}{\alpha^N+\beta^N}\right| \approx 0.000166$.

This shows that this $\beta^N$ is a good approximation for $\alpha^N + \beta^N$ when $N$ is large.