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COS 488/MAT 474
Problem Set 3, Q3

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AofA Exercise 4.38 Calculate an asymptotic expansion for $(3N)!/(N!)^3$.

Solution. We use Stirling's formula for factorials:

$$(3N)! = \sqrt{6\pi N}\left(\frac{3N}{e}\right)^{3N}\left(1+O\left(\frac{1}{N}\right)\right)$$

$$(N!)^3 = (\sqrt{2\pi N})^3\left(\frac{N}{e}\right)^{3N}\left(1+O\left(\frac{1}{N}\right)\right)^3$$

Note that $\left(1+O\left(\frac{1}{N}\right)\right)^k = \left(1+O\left(\frac{1}{N}\right)\right)$ for all $k \geq 1$ (for the reasoning given on p. 172 + induction). Therefore, we have

$$\begin{aligned}\frac{(3N)!}{(N!)^3} &= \frac{\sqrt{3}\cdot\sqrt{2\pi N}\left(\frac{3N}{e}\right)^{3N}\left(1+O\left(\frac{1}{N}\right)\right)}{(\sqrt{2\pi N})^3\left(\frac{N}{e}\right)^{3N}\left(1+O\left(\frac{1}{N}\right)\right)} \\ &= \frac{\sqrt{3}}{2\pi N}\cdot 3^{3N}\left(1+O\left(\frac{1}{N}\right)\right) \\ &= \frac{27^N\sqrt{3}}{2\pi N}\left(1+O\left(\frac{1}{N}\right)\right).\end{aligned}$$