5/5, well written and shows a clear concept-level understanding of the proof of Theorem 4.8. Great work!

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COS 488/MAT 474
Problem Set 3, Q4

AofA Exercise 4.71 Show that

$$P(N) = \sum_{0\le k<N} \frac{(N-k)!(N-k)^k}{N!} = \sqrt{\pi N/2} + O(1).$$

Solution.

$$\begin{aligned}
\frac{(N-k)!(N-k)^k}{N!} &= \frac{(N-k)^k}{N(N-1)\cdots(N-k+1)} \\
&= \frac{1}{1+\frac{k}{N-k}}\cdot\frac{1}{1+\frac{k-1}{N-k}}\cdots\frac{1}{1+\frac{2}{N-k}}\cdot\frac{1}{1+\frac{1}{N-k}} \\
&= \exp\left(-\sum_{j=1}^{k}\ln(1+\tfrac{j}{N-k})\right) \\
&= \exp\left(-\sum_{j=1}^{k}\left(\tfrac{j}{N-k}+O(\tfrac{j^2}{(N-k)^2})\right)\right) \\
&= \exp\left(-\frac{k(k+1)}{2(N-k)}+O\left(\frac{k^3}{(N-k)^2}\right)\right) \\
&= e^{\frac{-k^2}{2(N-k)}}\left(1+O\left(\frac{k}{N-k}\right)+O\left(\frac{k^3}{(N-k)^2}\right)\right) \qquad \text{when } k=o(N^{2/3}).
\end{aligned}$$

To simplify this, first we simplify the error terms using the Taylor expansion for $\frac{1}{1-x}$:

$$\frac{k}{N-k} = \frac{k}{N}\cdot\frac{1}{1-\frac{k}{N}} = \frac{k}{N}\left(1+\frac{k}{N}+O(\tfrac{k^2}{N^2})\right) = O(\tfrac{k}{N}), \qquad \text{therefore } O(\tfrac{k}{N-k}) \to O(\tfrac{k}{N}).$$

(This works because $k=o(N^{2/3})$.) Similarly,

$$\frac{k^3}{(N-k)^2} = \frac{k^3}{N^2}\cdot\left(\frac{1}{1-\frac{k}{N}}\right)^2 = \frac{k^3}{N^2}\left(1+2\frac{k}{N}+O(\frac{k^2}{N^2})\right) = O(\tfrac{k^3}{N^2}), \qquad \text{therefore } O(\tfrac{k^3}{(N-k)^2}) \to O(\tfrac{k^3}{N^2}).$$

For the leading term, we will compare it to $e^{-k^2/(2N)}$:

$$\begin{aligned}
\frac{e^{-k^2/(2N+2K)}}{e^{-k^2/(2N)}} &= \exp\left(\frac{k^2}{2N} - \frac{k^2}{2N-2k}\right) \\
&= \exp\left(\frac{-k^3}{2(N^2-Nk)}\right) \\
&= \exp\left(\frac{-(\frac{k}{N^{2/3}} \cdot N^{2/3})^3}{2(N^2-Nk)}\right) \\
&= \exp\left(-\left(\frac{k}{N^{2/3}}\right)^3 \cdot \frac{N^2}{2(N^2-Nk)}\right) \\
&\to e^0 = 1 \qquad \text{as } N \to \infty, \text{ because } k = o(N^{2/3}).
\end{aligned}$$

Therefore we have $e^{-k^2/(2N+2K)} \sim e^{-k^2/2N}$ as $N \to \infty$.

Thus, we have the following relative approximation which holds for $k = o(N^{2/3}$:

$$\frac{(N-k)!(N-k)^k}{N!} = e^{-k^2/(2N)}\left(1 + O\left(\frac{k}{N}\right) + O\left(\frac{k^3}{N^2}\right)\right).$$

To approximate $P(N) = \sum_{0 \le k < N} \frac{(N-k)!(N-k)^k}{N!}$, we define k_0 to be an integer that is $o(N^{2/3})$ and split the sum into two parts (as in the proof of Theorem 4.8). When $k_0 < k \le N$, the terms are all exponentially small. So we can write

$$P(N) = 1 + \sum_{1 \le k \le k_0} e^{-k^2/(2N)}\left(1 + O\left(\frac{k}{N}\right) + O\left(\frac{k^3}{N^2}\right)\right) + \Delta,$$

where Δ represents a term that is exponentially small.

The rest of the analysis proceeds identically to the analysis of the Q-function in the proof of Theorem 4.8. Therefore, the conclusion is the same as that of Theorem 4.8:

$$P(N) = \sqrt{\pi N/2} + O(1).$$