

AofA Exercise 4.23 Give an asymptotic expansion for $\frac{N}{N-1} \ln \frac{N}{N-1}$.

Solution.

$$\begin{aligned} \frac{N}{N-1} \ln \frac{N}{N-1} &= \frac{1}{1 - \frac{1}{N}} \ln \frac{1}{1 - \frac{1}{N}} \\ &= \sum_{k=1}^{\infty} H_k \left(\frac{1}{N} \right)^k \quad \left(\text{where } H_k = \sum_{j=1}^k \frac{1}{j} \text{ is the } k\text{th harmonic number; see p. 93} \right) \\ &= \frac{1}{N} + \frac{3}{2N^2} + \frac{11}{6N^3} + O\left(\frac{1}{N^4}\right). \end{aligned}$$