

COS 488 - Homework 3 - Question 3

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Let $f(N) = \ln \frac{(3N)!}{(N!)^3} = \ln((3N)!) - 3 \ln(N!)$. Then, by Stirling's approximation, we have that

$$\begin{aligned} f(N) &= 3N \ln(3N) - 3N + \ln \sqrt{6\pi N} + O\left(\frac{1}{N}\right) \\ &\quad - 3\left(N \ln(N) - N + \ln \sqrt{2\pi N} + O\left(\frac{1}{N}\right)\right) \\ &= 3N \ln 3 + \ln \frac{(6\pi N)^{1/2}}{(2\pi N)^{3/2}} + O\left(\frac{1}{N}\right) \\ &= 3N \ln 3 + \ln \sqrt{3} - \ln(2\pi N) + O\left(\frac{1}{N}\right) \\ &= \left(3N + \frac{1}{2}\right) \ln 3 - \ln(2\pi N) + O\left(\frac{1}{N}\right), \end{aligned}$$

so

$$\frac{(3N)!}{(N!)^3} = e^{f(N)} = \frac{3^{3N+1/2}}{2\pi N} \left(1 + O\left(\frac{1}{N}\right)\right).$$