4/5

COS 488 - Homework 3 - Question 4

Matt Tyler

Let $q(N,k) = \ln \frac{(N-k)^k (N-k)!}{N!} = k\ln(N-k) + \ln((N-k)!) - \ln(N!)$. Then, by Stirling's approximation, we have that

$$\begin{aligned}
q(N,k) &= \left(N - k + \frac{1}{2}\right)\ln(N-k) - (N-k) + \ln\sqrt{2\pi} \\
&\quad - \left(\left(N + \frac{1}{2}\right)\ln(N) - N + \ln\sqrt{2\pi}\right) + k\ln(N-k) + O\left(\frac{1}{N}\right) \\
&= \left(N - k + \frac{1}{2}\right)\left(\ln\left(1 - \frac{k}{N}\right) + \ln(N)\right) - \left(N + \frac{1}{2}\right)\ln(N) + k + k\ln(N-k) + O\left(\frac{1}{N}\right) \\
&= \left(N - k + \frac{1}{2}\right)\ln\left(1 - \frac{k}{N}\right) - k\ln(N) + k + k\ln(N-k) + O\left(\frac{1}{N}\right) \\
&= \left(N + \frac{1}{2}\right)\ln\left(1 - \frac{k}{N}\right) + k + O\left(\frac{1}{N}\right).
\end{aligned}$$

Then, using the asymptotic expansion $\ln\left(1 - \frac{k}{N}\right) = -\frac{k}{N} - \frac{k^2}{2N^2} + O\left(\frac{k^3}{N^3}\right)$, we have that

$$\begin{aligned}
q(N,k) &= \left(N + \frac{1}{2}\right)\ln\left(1 - \frac{k}{N}\right) + k + O\left(\frac{1}{N}\right) \\
&= \left(N + \frac{1}{2}\right)\left(-\frac{k}{N} - \frac{k^2}{2N^2} + O\left(\frac{k^3}{N^3}\right)\right) + k + O\left(\frac{1}{N}\right) \\
&= -k - \frac{k^2}{2N} + O\left(\frac{k^3}{N^2}\right) - \frac{k}{2N} - \frac{k^2}{4N^2} + O\left(\frac{k^3}{N^3}\right) + k + O\left(\frac{1}{N}\right) \\
&= \frac{-k^2}{2N} + O\left(\frac{k^3}{N^2}\right) + O\left(\frac{k}{N}\right).
\end{aligned}$$

Therefore,

$$\frac{(N-k)^k (N-k)!}{N!} = e^{q(N,k)} = e^{\frac{-k^2}{2N}}\left(1 + O\left(\frac{k^3}{N^2}\right) + O\left(\frac{k}{N}\right)\right).$$

-1pt, this is only follows from the last step for k << N^2/3,

Then, since $\frac{N!}{(N-k)!N^k}$ also has this same asymptotic expansion and

$$\sum_{k=0}^{N} \frac{N!}{(N-k)!N^k} = \sqrt{\frac{\pi N}{2}} + O(1)$$

(and the only information about $\frac{N!}{(N-k)!N^k}$ that we used in this derivation was that it had the asymptotic expansion given above), we also have that

$$\sum_{k=0}^{N} \frac{(N-k)^k (N-k)!}{N!} = \sqrt{\frac{\pi N}{2}} + O(1),$$

as desired.