COS 488 Problem Set #3 Question #3

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$$N! = \sqrt{2\pi N}\left(\frac{N}{e}\right)^N \left(1 + \frac{1}{12N} + \frac{1}{288N^2} + O\left(\frac{1}{N^3}\right)\right)$$

$$\frac{(3N)!}{N!^3} = \frac{\sqrt{6\pi N}\left(\frac{3N}{e}\right)^{3N}\left(1 + \frac{1}{36N} + \frac{1}{2592N^2} + O\left(\frac{1}{N^3}\right)\right)}{(2\pi N)^{3/2}(N/e)^{3N}\left(1 + \frac{1}{12N} + \frac{1}{288N^2} + O\left(\frac{1}{N^3}\right)\right)^3}$$

This gives a primary term of $\frac{\sqrt{3}}{2\pi N}3^{3N}$ and then a relative error term. Note, first, that

$$\left(1 + \frac{1}{12N} + \frac{1}{288N^2} + O\left(\frac{1}{N^3}\right)\right)^3 = 1 + \frac{1}{4N} + \frac{1}{48N^2} + \frac{1}{96N^2} + O\left(\frac{1}{N^3}\right) = 1 + \frac{1}{4N} + \frac{1}{32N^2} + O\left(\frac{1}{N^3}\right)$$

As a result, the relative error term is given by

$$\begin{aligned}
\frac{1 + \frac{1}{36N} + \frac{1}{2592N^2} + O(1/N^3)}{1 + \frac{1}{4N} + \frac{1}{32N^2} + O(1/N^3)} &= \left(1 + \frac{1}{36N} + \frac{1}{2592N^2} + O\left(\frac{1}{N^3}\right)\right)\left(1 - \left(\frac{1}{4N} + \frac{1}{32N^2}\right) + \left(\frac{1}{4N} + \frac{1}{32N^2}\right)^2 + O\left(\frac{1}{N^3}\right)\right) \\
&= \left(1 + \frac{1}{36N} + \frac{1}{2592N^2} + O\left(\frac{1}{N^3}\right)\right)\left(1 - \frac{1}{4N} - \frac{1}{32N^2} + \frac{1}{16N^2} + O\left(\frac{1}{N^3}\right)\right) \\
&= \left(1 + \frac{1}{36N} + \frac{1}{2592N^2} + O\left(\frac{1}{N^3}\right)\right)\left(1 - \frac{1}{4N} + \frac{1}{32N^2} + O\left(\frac{1}{N^3}\right)\right) \\
&= 1 - \frac{1}{4N} + \frac{1}{36N} - \frac{1}{144N^2} + \frac{1}{32N^2} + \frac{1}{2592N^2} + O\left(\frac{1}{N^3}\right) \\
&= 1 - \frac{2}{9N} + \frac{2}{81N^2} + O\left(\frac{1}{N^3}\right)
\end{aligned}$$

Hence,

$$\frac{(3N)!}{N!^3} = \frac{\sqrt{3}}{2\pi N}3^{3N}\left(1 - \frac{2}{9N} + \frac{2}{81N^2} + O\left(\frac{1}{N^3}\right)\right)$$

Nice work!