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COS 488 Problem Set #3 Question #4

Tim Ratigan

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$$\sum_{k=0}^{N-1}\frac{(N-k)^k(N-k)!}{N!}=\sum_{k=0}^{N-1}\prod_{j=0}^{k-1}\left(\frac{N-k}{N-j}\right)$$
$$=\sum_{k=0}^{N-1}\exp\left(-\sum_{j=0}^{k-1}\log\frac{N-j}{N-k}\right)$$
$$=\sum_{k=0}^{N-1}\exp\left(-\sum_{j=1}^{k}\log\frac{N-k+j}{N-k}\right)$$
$$=\sum_{k=0}^{N-1}\exp\left(-\sum_{j=1}^{k}\log\left(1+\frac{j}{N-k}\right)\right)$$
$$=\sum_{k=0}^{N-1}\exp\left(-\sum_{j=1}^{k}\left(\frac{j}{N-k}+O\left(\frac{j^2}{(N-k)^2}\right)\right)\right)$$
$$=\sum_{k=0}^{N-1}\exp\left(-\frac{k(k+1)}{2(N-k)}+O\left(\frac{k^3}{(N-k)^2}\right)\right)$$

Note that when $k=o(N^{2/3})$, we have $\frac{1}{N-k}=\frac{1}{N}\frac{1}{1-k/N}=\frac{1}{N}(1+O(k/N))=\frac{1}{N}+O\left(\frac{k}{N^2}\right)$. As a result,

$$\frac{k^3}{(N-k)^2}=\frac{k^3}{N^2}+O\left(\frac{k^4}{N^3}\right)$$

k^4/N^3 is O(k^3/N^2) for all k, so the O(k^4/N^3) term is unnecessary

$$\frac{k(k+1)}{2(N-k)}=\frac{k^2}{2(N-k)}+O\left(\frac{1}{N}\right)=\frac{k^2}{2}\left(\frac{1}{N}+O\left(\frac{k}{N^2}\right)\right)+O\left(\frac{1}{N}\right)=\frac{k^2}{2N}+O\left(\frac{k^3}{N^2}\right)+O\left(\frac{1}{N}\right)$$

As a result, we have that the exponential reduces to $\exp\left(-\frac{k^2}{2N}+O\left(\frac{1}{N}\right)+O\left(\frac{k^3}{N^2}\right)+O\left(\frac{k^4}{N^3}\right)\right)$. Since $\prod_{j=0}^{k-1}\left(\frac{N-k}{N-j}\right)$ is monotonically decreasing and exponential at $k_0=o(N^{2/3})$ by our approximation, it is exponential for $k>k_0$ and so we can approximate $\sum_{k=k_0+1}^{N-1}\frac{(N_k)^k(N-k)!}{N!}$ by $\sum_{k=k_0}^{\infty}\exp\left(-\frac{k^2}{2N}\right)$ to within $O(1)$. As a result, we have

$$\sum_{k=0}^{N-1}\frac{(N-k)^k(N-k)!}{N!}=\sum_{k=0}^{k_0}e^{-k^2/2N}e^{O\left(\frac{1}{N}\right)+O\left(\frac{k^3}{N^2}\right)+O\left(\frac{k^4}{N^3}\right)}+\sum_{k=k_0}^{\infty}+O(1)$$
$$=\sum_{k=0}^{k_0}e^{-k^2/2N}\left(1+O\left(\frac{1}{N}\right)+O\left(\frac{k^3}{N^2}\right)+O\left(\frac{k^4}{N^3}\right)\right)+\sum_{k=k_0}^{\infty}e^{-k^2/2N}+O(1)$$

First of all, note that $e^{-k^2/2N}\leq 1$ so the $O\left(\frac{1}{N}\right)$ contributes $O\left(\frac{k_0}{N}\right)=o(N^{-1/3})$ in absolute terms. For $k=o(N^{2/3})$, $O\left(\frac{k^4}{N^3}\right)=O\left(\frac{k^3}{N^2}\right)$ so it suffices to consider the contribution of this error term. For $k=O(N^{1/2})$, it suffices to note that $e^{-k^2/2N}\leq 1$ and so $\sum_{k=0}^{O(N^{1/2})}O\left(\frac{k^3}{N^2}\right)=O(1)$, while for $k>O(N^{1/2})$ we have $e^{-k^2/2N}$ is exponential in N so the absolute contribution of the relative error is easily $O(1)$. It then follows that:

$$\begin{aligned}
\sum_{k=0}^{N-1} \frac{(N-k)^k (N-k)!}{N!} &= \sum_{k \geq 0} e^{-k^2/2N} + O(1) \\
&= \sqrt{N} \int_0^\infty e^{-x^2/2N} dx + O(1) \\
&= \sqrt{\pi N/2} + O(1)
\end{aligned}$$