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Exercise 5.7

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Derive an EGF for the number of permutations whose cycles are all of odd length.

We have class D, the class of all permutations whose cycles are only of odd length, and the notion of size being $|d|$, the number of atoms in d where an atom Z has size 1 and GF of z. We have an exponential generating function of

$$D(z) = \sum_{d \in D} \frac{z^{|d|}}{|d|!} = \sum_{N \geq 0} D_N \frac{z^N}{N!}$$

and the following construction: D is the set of all cycles that are odd.

$$D = SET(CYC_{odd}(Z))$$

This yields the below GF, where $2k + 1$ is the kth odd number starting with $k = 0$.

$$D(z) = exp(\frac{z}{1} + \frac{z^3}{3} + \frac{z^5}{5} + ...) = exp(\sum_{k \geq 0} \frac{z^{2k+1}}{2k+1})$$

$$D(z) = exp(\sum_{k \geq 0} \frac{z^k}{k}) / exp(\sum_{k \geq 0} \frac{z^{2k}}{2k})$$

$$= \frac{1}{1-z}\sqrt{1-z^2} = \sqrt{\frac{1-z^2}{(1-z)(1-z)}}$$

$$= \frac{\sqrt{1+z}}{\sqrt{1-z}}$$