5/5

David Luo
Exercise 5.23

Show that the probability that all of the cycles are of odd length in a random permutation of length N is $1/\sqrt{\pi N/2}$.

From Exercise 5.7, we showed that the generating function for the number of permutations whose cycles are all of odd length is

$$D(z) = \frac{\sqrt{1+z}}{\sqrt{1-z}}$$

Then apply the following corollary to solve: If $f(z)$ has radius of convergence $> \rho$ with $f(\rho) \neq 0$ then

$$[z^n]\frac{f(z)}{(1-z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)}\rho^n n^{\alpha-1}$$

We have $\rho = 1$, $\alpha = 1/2$, and $f(z) =$ the numerator, so the number of N-length permutations whose cycles are odd in length is

$$[z^N]D(z) \sim N!\frac{\sqrt{2}}{\Gamma(1/2)}1^N N^{-\frac{1}{2}} = N!\frac{\sqrt{2}}{\sqrt{\pi N}}$$

$$= \frac{N!}{\sqrt{\pi N/2}}$$

And from lecture, we have a proof that shows that the total number of permutations of length N is $N!$. Simply divide number of odds by number of permutations in total to get the desired $1/\sqrt{\pi N/2}$.