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COS 488
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PS4 - Q2

We can define two atoms, one for external nodes and one for internal nodes. Let's denote them Z_1, Z_2 respectively. Since we count both internal and external nodes, they will each have size 1 and GF z. Then our construction for binary trees still holds as usual: A binary tree is an external node or an internal node connected to two binary trees. Letting U be the class of binary trees, we have:

$$U = Z_1 + U \times Z_2 \times U$$

which gives the OGF equation

$$U(z) = z + zU(z)^2$$

Solving this equation for U(z) with the quadratic formula gives:

$$zU(z) = \frac{1}{2}(1 \pm \sqrt{1 - 4z^2})$$

-1pt, explain why the minus sign is taken

Which we can expand with the binomial theorem to

$$zU(z) = -\frac{1}{2}\sum_{N \geq 1}\binom{\frac{1}{2}}{N}(-4z^2)^N = -\frac{1}{2}\sum_{N \geq 1}\binom{\frac{1}{2}}{N}(-4)^N z^{2N}$$

Now note that since the right hand size has only the even powers of z, the resulting powers of z that show up in U(z) are all the odd powers, $N = 1, 3, 5, \ldots$.

Setting the coefficients equal for any of these values and writing the power of z as 2N+1, for $N \geq 0$, we get:

$$\begin{aligned} U_{2N+1} &= -\frac{1}{2}\binom{\frac{1}{2}}{N+1}(-4)^{N+1} \\ &= -\frac{1}{2}\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots(\frac{1}{2}-N)(-4)^{N+1}}{(N+1)!} \\ &= \frac{1 * 3 * 5 \ldots (2N-1) * 2^N}{(N+1)!} \\ &= \frac{1}{N+1}\frac{1 * 3 * 5 \ldots (2N-1)}{N!}\frac{2 * 4 * 6 \ldots 2N}{1 * 2 * 3 \ldots N} \\ &= \frac{1}{N+1}\binom{2N}{N} \end{aligned}$$

Appreciate the rederivation of the Catalan sequence, but in general when a problem asks for an explicit form for U(z) it's fine to leave it not as a power series so you could've stopped after using the quadratic formula. The power series form is required only if [z^n]U(z) is asked for.

which is precisely the Catalan coefficient for N. We also have that the coefficient is 0 for all even powers of z, since they do not show up in U(z). Hence, we have a general expression for U(z), which is:

$$U(z) = \sum_{N \geq 0}\frac{1}{N+1}\binom{2N}{N}z^{2N+1}$$

We can check that this matches up with the first couple terms of the explicit sequence given,

$$U(z) = z + z^3 + 2z^5 + 5z^7 + 14z^9 + ...$$

The first 5 coefficients here are exactly the first five numbers in the Catalan sequence.