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COS 488 Week 4: Q2

Dylan Mavrides

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5.3) Let $\mathcal{U}$ be the set of binary trees with the size of a tree defined to be the total number of nodes (internal plus external), so that the generating function for its counting sequence is $U(z) = z + z^3 + 2z^5 + 5z^7 + 14z^9 + ...$ Derive an explicit expression for $U(z)$.

I have found two ways to do this problem, and will do the easier way first, followed by the way that I think was intended.

First, we recognize that we can begin with the sequence/generating function for the catalan numbers:

Sequence: 1, 1, 2, 5, 7, 14, ...

Function:

$$1 + z + 2z^2 + 5z^3 + 14z^4 + ... = \sum_{N \geq 0} \frac{1}{N+1}\binom{2N}{N} z^N = \frac{1}{2z}(1 - \sqrt{1-4z})$$

Now we use substitution and replace z with z^2. After doing this, we then multiply the expression by z to right-shift, giving:

$$U(z) = z + z^3 + 2z^5 + 5z^7 + 14z^9 + ... = \sum_{N \geq 1} \frac{1}{N}\binom{2(N-1)}{N-1} z^{2N-1} = \frac{1}{2z}(1 - \sqrt{1-4z^2})$$

The second solution notes that we begin with the sequence of trees, counting both the internal and external nodes, and including the case with just one external node. (i.e. we have one external node, then one internal node two external nodes, then two internal nodes, three external nodes, etc. as illustrated on slide 8 of AA05-AC.pdf)

We node that this situation describes the desired situation, since when we have 5 total nodes, we have two possible trees, when we have 7, there are 5 trees, and so on. We now represent this symbolically as: $U = z + z \times U \times U$ (or in English: a binary tree as described is either a single node, or a node with two subtrees contained in U). This gives a quadratic equation that we can solve for U in terms of z using the quadratic equation, giving:

$$U(z) = \frac{1 \pm \sqrt{1-4z^2}}{2z}$$

Now we note that we can use Taylor's theorem to expand the first term of the equation (i.e. we get the first derivative and plug in zero) for the plus and minus cases. The derivative is:

$$\pm\frac{(1 - 1/\sqrt{1-4z^2})}{2z^2}$$

Now if you apply l'hôpital's rule twice and plug in zero, for the + case we get ∞, the - case gives 1, thus we must use the minus case, as we know this coefficient to be 1. (From here, if you want the coefficients, they are obtained by the reasoning I gave above with the first solution.)
(Worked with Maryam B.)