Homework 4: Exercise 5.1

Maryam Bahrani (mbahrani)

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Let $\mathcal{B}_{000}$ be the class of bit strings, with the size function defined as the number of bits. Let $B(z)$ be its generating function. Let $\mathcal{Z}_0$ denote a zero bit and $\mathcal{Z}_1$ a one bit.

This class can be specified as follows

$$\mathcal{B}_{000} = E + \mathcal{Z}_0 + \mathcal{Z}_0 \times \mathcal{Z}_0 + (\mathcal{Z}_1 + \mathcal{Z}_0 \times \mathcal{Z}_1 + \mathcal{Z}_0 \times \mathcal{Z}_0 \times \mathcal{Z}_1) \times \mathcal{B}_{000}$$

The symbolic then gives the following generating function equation

$$B(z) = 1 + z + z^2 + (z + z^2 + z^3)B(z)$$
$$B(z) = \frac{1 + z + z^2}{1 - z - z^2 - z^3}.$$

Let $f(z)$ be numerator and $g(z)$ the denominator. Finally, we can apply the rational functions transfer theorem from Lecture 4, produced below, to derive asymptotics.

Theorem 1 *Assume that a rational GF $f(z)/g(z)$ with $f(z)/g(z)$ with $f(z)$ and $g(z)$ relatively prime and $g(0) = 0$ has a unique pole $1/\beta$ of smallest modulus and that the multiplicity of β is ν. Then,*

$$[z^n]\frac{f(z)}{g(z)} \sim C\beta^n n^{\nu-1} \text{ where } C = \nu\frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)}$$

We can easily check that f and g are relatively prime, since -1 is a root for f but not g. The roots of g and the corresponding moduli can be computed using a calculator:

$$z_1 \approx 0.54369, \qquad |z_1| \approx 0.54369$$
$$z_2 \approx -0.77184 - 1.11514i, \qquad |z_2| \approx 1.3562$$
$$z_3 \approx -0.77184 + 1.11514i, \qquad |z_3| \approx 1.3562$$

Therefore, there is a unique pole of smallest modulus $1/\beta \approx 0.54369 \implies \beta \approx 1.83928$, with multiplicity $\nu = 1$. Therefore, C can be computed as

$$C = \nu\frac{(-\beta)^\nu f(1/\beta)}{g^{(\nu)}(1/\beta)} = \frac{-\beta f(1/\beta)}{g'(1/\beta)} = \frac{-\beta(1 + 1/\beta + (1/\beta)^2)}{-1 - 2(1/\beta) - 3(1/\beta)^2}$$
$$= \frac{-1.83928(1 + 1/1.83928 + 1/1.83928^2)}{-1 - 2(1/1.83928) - 3(1/1.83928)^2} \approx 1.13745.$$

The theorem thus gives

$$[z^n]B(z) = [z^n]\frac{f(z)}{g(z)} \sim 1.13745(1.83928)^n.$$