Homework 4: Exercise 5.3

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We wish to derive an explicit expression for $U(z)$, which has the following expansion:

$$U(z) = z + z^3 + 2z^5 + 5z^7 + 14z^9 + \cdots .$$

We observe the similarity between this counting sequence and the Catalan numbers:

$$C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots .$$
$$zC(z^2) = z + z^3 + 2z^5 + 5z^7 + 14z^9 + \cdots = U(z)$$

Given that the closed form for the generating function for Catalan numbers is well-known, we have

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$
$$U(z) = zC(z^2) = \frac{1 - \sqrt{1 - 4z^2}}{2z}.$$

Furthermore, the closed form for the coefficients of $U(z)$ can be derived from the Catalan numbers:

$$[z^n]C(z) = \frac{1}{n+1}\binom{2n}{n}$$
$$[z^n]U(z) = [z^n]zC(z^2)$$

Therefore,

$$[z^n]U(z) = \begin{cases} 0 & n \text{ even} \\ [z^{\frac{n-1}{2}}]C(z) = \frac{1}{\frac{n+1}{2}}\binom{n-1}{\frac{n-1}{2}} = \frac{2}{n+1}\binom{n-1}{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$