Homework 4: Exercise 5.23

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In problem 5.7, we derived the following EGF for permutations with odd-length cycles:

$$P(z) = \sqrt{\frac{1+z}{1-z}}.$$

We will use the Radius-of-Convergence transfer theorem from lecture 5, reproduced below, to derive asymptotics for $P(z)$:

Corollary 1 *If $f(z)$ has radius of convergence $> \rho$ with $f(\rho) \neq 0$, then*

$$[z^n]\frac{f(z)}{(1-z/\rho)^{\alpha}} \sim \frac{f(\rho)}{\Gamma(\alpha)}\rho^{-n}n^{\alpha-1}.$$

In this case, we have $f(z) = \sqrt{1+z}$, $g(z) = \sqrt{1-z}$, $\alpha = 1/2$ and $\rho = 1$. The numerator has an infinite radius of convergence, and $f(1) = \sqrt{2} \neq 0$. Therefore,

$$[z^n]\sqrt{\frac{1+z}{1-z}} \sim \frac{f(1)}{\Gamma(1/2)}1^{-n}n^{\frac{1}{2}-1} = \frac{\sqrt{2}}{\sqrt{n\pi}} = \frac{1}{\sqrt{n\pi/2}}.$$

Note that the *number* of permutations of size n with odd-length cycles is $n!$ times the coefficient of z^n (to adjust for the division by $n!$ in the definition of EGFs). Therefore, the probability that a random permutation consists of only odd cycles is

$$\frac{n!\frac{1}{\sqrt{n\pi/2}}}{n!} = \frac{1}{\sqrt{n\pi/2}}$$

as desired.