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COS 488/MAT 474
Problem Set 4, Q1

4.5/5

AofA Exercise 5.1 How many bitstrings of length N have no 000?

Solution. A binary string with no occurrence of 000 is either empty; 0; 00; or 1, 01, or 001 followed by a binary string with no 000. In symbols, we can write this as

$$\mathcal{B}_{000} = \epsilon + \mathcal{Z}_0 + (\mathcal{Z}_0 \times \mathcal{Z}_0) + (\mathcal{Z}_1 + (\mathcal{Z}_0 \times \mathcal{Z}_1) + (\mathcal{Z}_0 \times \mathcal{Z}_0 \times \mathcal{Z}_1)) \times \mathcal{B}_{000}.$$

By the transfer theorem, we can translate this into a generating function:

$$B_{000}(z) = 1 + z + z^2 + (z + z^2 + z^3)B_{000}(z)$$
$$\Rightarrow B_{000}(z) = \frac{1 + z + z^2}{1 - z - z^2 - z^3}.$$

We can now find asymptotic approximations for the coefficients of this generating function. By Theorem 4.1, we have

$$[z^n]\frac{1 + z + z^2}{1 - z - z^2 - z^3} \sim C\beta^n,$$

where

$$\frac{1}{\beta} = \frac{1}{3}\left(\sqrt[3]{17 + 3\sqrt{33}} - \frac{2}{\sqrt[3]{17 + 3\sqrt{33}}}\right) - \frac{1}{3} \approx 0.54369,$$

the unique root of $1 - z - z^2 - z^3$ of smallest modulus (so $\beta \approx 1.839$), and

$$C = \frac{\beta(1 + \frac{1}{\beta} + \frac{1}{\beta^2})}{1 + \frac{2}{\beta} + \frac{3}{\beta^2}} \approx 0.228.$$

Should come out to ~1.137

-0.5