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COS 488/MAT 474
Problem Set 4, Q2

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AofA Exercise 5.3 Let $\mathcal{U}$ be the set of binary trees with the size of a tree defined to be the total number of nodes (internal plus external). Derive an explicit expression for the generating function $U(z)$.

Solution. A binary tree is either a node, or a node connected to two binary trees. In symbols, we can write this as

$$\mathcal{U} = \mathcal{Z} + \mathcal{U} \times \mathcal{Z} \times \mathcal{U}.$$

There is only one type of atom (the node, whise class is denoted by $\mathcal{Z}$), and the size of each node is 1. So the generating function for $\mathcal{Z}$ is z. By the transfer theorem, we can write the following functional equation for the generating function of $\mathcal{U}$:

$$U(z) = z + z(U(z))^2.$$

We solve this equation using the quadratic formula to obtain:

$$U(z) = \frac{1 \pm \sqrt{1 - 4z^2}}{2z}.$$

If we expand the Taylor series for $\sqrt{1-4z} = 1 - 2z - 2z^2 - \cdots$, we see that we must choose the minus sign in order for the coefficients of $U(z)$ to be positive. Therefore, our final answer is

$$U(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z}.$$