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COS 488/MAT 474
Problem Set 4, Q3

5/5

AofA Exercise 5.7 Derive an EGF for the number of permutations whose cycles are all of odd length.

Solution. The class we're working with, which we'll call $\mathcal{A}$, is the class of all sets of odd-length cycles of atoms. In symbols, we can write this as

$$\mathcal{A} = SET\left(\sum_{k=1}^{\infty} CYC_k(\mathcal{Z}) - \sum_{k=1}^{\infty} CYC_{2k}(\mathcal{Z})\right).$$

By the transfer theorem, we can translate this into an exponential generating function for $\mathcal{A}$:

$$\begin{aligned}
A(z) &= \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} - \sum_{k=1}^{\infty} \frac{z^{2k}}{2k}\right) \\
&= \exp\left(\ln \frac{1}{1-z} - \frac{1}{2} \ln \frac{1}{1-z^2}\right) \\
&= \exp\left(\ln\left(\frac{\sqrt{1-z^2}}{1-z}\right)\right) \\
&= \frac{\sqrt{(1+z)(1-z)}}{\sqrt{(1-z)^2}} \\
&= \sqrt{\frac{1+z}{1-z}}.
\end{aligned}$$