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COS 488/MAT 474
Problem Set 4, Q4

5/5

AofA Exercise 5.23 Show that the probability that all the cycles are of odd length in a random permutation of length N is asymptotic to $\frac{1}{\sqrt{\pi N/2}}$.

Solution. Recall from Exercise 5.7 that the EGF for permutations with all cycles of odd length is

$$A(z) = \sqrt{\frac{1+z}{1-z}}.$$

We can apply the radius-of-convergence transfer theorem with $f(z) = \sqrt{1+z}$ and $\alpha = \frac{1}{2}$ to obtain the following asymptotic approximation for the coefficients of $A(z)$:

$$[z^N]A(z) \sim \frac{\sqrt{2}}{\Gamma(\frac{1}{2})} N^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi N/2}}.$$

The total number of permutations of length N is $N!$, so the coefficient of z^N in the EGF for permutations, $P(z)$, is $\frac{N!}{N!} = 1$. Therefore, the probability that a random permutation of length N will have all odd cycles is asymptotic to

$$\frac{[z^N]A(z)}{[z^N]P(z)} = \frac{\frac{1}{\sqrt{\pi N/2}}}{1} = \frac{1}{\sqrt{\pi N/2}}.$$