COS 488 - Homework 4 - Question 1

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Let Z_0 denote the atom of one 0-bit, and let Z_1 denote the atom of one 1-bit. Let B_{000} denote the class of binary strings with no 00 (with $|b|$ denoting the number of bits in b for each $b \in B_{000}$, and let E denote the empty string. Then, an element of B_{000} is either empty, a single 0-bit, a sequence of two 0-bits, or an element of B_{000} preceded by 1, 01, or 001. Therefore, we have the construction

$$B_{000} = E + Z_0 + Z_0 \times Z_0 + (Z_1 + Z_0 \times Z_1 + Z_0 \times Z_0 \times Z_1) \times B_{000},$$

which gives the OGF equation

$$B_{000}(z) = 1 + z + z^2 + (z + z^2 + z^3)B_{000}(z)$$

whose solutions is the equation

$$B_{000}(z) = \frac{1 + z + z^2}{1 - z - z^2 - z^3}.$$

The expression $1 - z - z^2 - z^3$ has the following three roots:

$$\alpha = \frac{1}{3}\left(\sqrt[3]{17 + 3\sqrt{33}} - \frac{2}{\sqrt[3]{17 + 3\sqrt{33}}} - 1\right)$$

$$\beta = \frac{1}{3}\left(\frac{(-1 + i\sqrt{3})\sqrt[3]{17 + 3\sqrt{33}}}{2} + \frac{1 + i\sqrt{3}}{\sqrt[3]{17 + 3\sqrt{33}}} - 1\right)$$

$$\gamma = \frac{1}{3}\left(\frac{(-1 - i\sqrt{3})\sqrt[3]{17 + 3\sqrt{33}}}{2} + \frac{1 - i\sqrt{3}}{\sqrt[3]{17 + 3\sqrt{33}}} - 1\right)$$

Therefore, since

$$|\alpha| \approx 0.5437 < 1.3562 \approx |\beta| = |\gamma|,$$

we have by the rational functions transfer theorem with $f(z) = 1 + z + z^2$ and $g(z) = 1 - z - z^2 - z^3$ that

$$[z^n]B_{000}(z) \sim -\frac{f(\alpha)}{g'(\alpha)\alpha^{n+1}} \approx \frac{1.839}{(2.974)(0.5437^{n+1})} \approx (1.137)(1.839^n).$$

In other words, there are approximately $(1.137)(1.839^n)$ bitstrings of length n that do not contain 000.