COS 488 - Homework 4 - Question 2

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Let $\mathcal{U}$ denote the class of all binary trees with $|u|$ being the total number of nodes in u for each $u \in \mathcal{U}$. Let $Z_.$ denote the atom of one node, which has size 1. Then, an element of $\mathcal{U}$ is either a single node or a node connected to two other elements of $\mathcal{U}$, so we have the construction

$$\mathcal{U} = Z_. + \mathcal{U} \times Z_. \times \mathcal{U},$$

which gives the OGF equation

$$\mathcal{U}(z) = z + z\mathcal{U}(z)^2$$

whose solution, using the quadratic formula, is the equation

$$\mathcal{U}(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z}.$$

(We take the negative square root because $\lim_{z \to 0} U(z) = 0$.)
Since

$$\frac{1 - \sqrt{1 - 4z}}{2z} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{n+1},$$

$$U(z) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^{2n+1}}{n+1}.$$