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Analytic Combinatorics Homework 5 Problem 3

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We can think of an arrangement as an ordered pair consisting of a *set* of elements (which are not being permuted), and a *sequence* of elements. We thus can write arrangements as $SET(Z) \star SEQ(Z)$. The generating function for arrangements is thus $e^z \cdot \frac{1}{1-z}$.

We have

$$\frac{e^z}{1-z} = \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \dots\right)(1 + z + z^2 + z^3 + \dots) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!}.$$

Thus the number of arrangements of n elements is $n!\left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n!}\right)$. A combinatorial interpretation of this is: if the set that is chosen to *not* be sequenced has size k, it can be chosen in $\binom{n}{k}$ ways, and the remaining elements can be ordered in $(n-k)!$ ways, thus giving $(n-k)!\binom{n}{k} = \frac{n!}{(n-k)!k!}(n-k)! = \frac{n!}{k!}$ possible arrangements. Thus the $\frac{n!}{k!}$ term in the sum counts the number of arrangements where the set that is made into a sequence has size $n-k$.