5/5

COS 488 Week 5: Q1

Dylan Mavrides

March 10, 2017

6.6) What proportion of the forests with N nodes have no trees consisting of a single node? For $N = 1, 2, 3$, and 4, the answers are $0, 1/2, 2/5$, and $3/7$, respectively.
First we find the number of forests on N nodes that have no trees consisting of a single node.
We first let $G(z)$ be the generating function for general trees. Thus $G(z) - z$ is the generating function for general trees minus trees of one node. Thus a forest of trees which have no trees consisting of a single node is

$$F(z) = SEQ(G(z) - z) = \frac{1}{1 + z - G(z)}$$

Note that G(z) is the catalan numbers shifted back 1 (by the online slides) thus we have:

$$F(z) = \frac{1}{1 + z - zT(z)} = \frac{1}{1 - \frac{1-\sqrt{1-4z}}{2} + z} = \frac{2}{2z + 1 - \sqrt{1-4z}}$$

and now we rationalize, giving:

$$F(z) = \frac{4z + 2 - 2\sqrt{1-4z}}{(2z+1)^2 - 1 + 4z} = \frac{2z + 1 - \sqrt{1-4z}}{2z(z+2)}$$

We temporarily ignore the z in the denominator, noting that we can shift the sequence by 1 later. Thus we have:

$$F'(z) = \frac{2z+1}{2(z+2)} - \frac{\sqrt{1-4z}}{2(z+2)}$$

We note that the first term will be something with $(1/2)^2$, but the second one will be 4^n, so the first one is exponentially small relative to the second and can be ignored. Thus we can apply the corollary of the radius of convergence transfer theorem. (Note that the radius of convergence is 2, so it's greater than 1.) We have $\rho = 1/4$, $\alpha = -1/2$ and $f(z) = \frac{1}{2(z+2)}$. We note that $\Gamma(-1/2) = -2\sqrt{\pi}$ thus we have the asymptotic approximation:

$$zF(z) \ \frac{4^n}{9\sqrt{\pi n^3}}$$

and therefore (note that $(n-1)^3$ and n^3 are asymptotically the same) we have:

$$F(z) = 4^{n-1} 9\sqrt{\pi n^3}$$

Now to find the total number of forests of size n, we want the total number of general trees of size n + 1 (we have a root element added in to a forest to create a tree to biject), thus we have $T_{n-2}\ \frac{4^{n-2}}{\sqrt{\pi n^3}}$. Dividing this with our other equation, we are left with 4/9 as our answer.

Worked with Eric N, Matt T.

you can just cite result on forests from class