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COS 488 Week 5: Q4

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7.45) Find the CGF for the total number of inversions in all involutions of length N. Use this to find the average number of inversions in an involution.
To solve this problem, we are going to try to find a way to express the total number of inversions over an involution of size n in terms of involutions (possible of several smaller sizes). we want to find a bijection between each inversion and some set of involutions. To do this, we consider an existing involution with some number of inversions. We attempt to remove one involution by removing two elements that are in an inversion, but we also remove any elements that are in a cycle with either of these two elements, thus we now need to address possible double/under-counting.
In the first case, we have the situation where those two elements are in a cycle with each other, thus we remove no extra elements, so we have $\binom{n}{2}$ ways of choosing these elements.
In the second case, we may have to remove three elements, if one of the two original elements is in a cycle with another element. In this situation, we consider the number of additional inversions we remove, and correct accordingly. We note that we have $i < j < k$ as our three elements, each must be involved in an inversion, and there must be exactly one fixed element. Thus we see that i and k cannot be our fixed elements, because then they would not be involved in an inversion (since they would stay before/after everything that changes). Thus j must be the fixed element, and there are two pairs (i, j) and (j, k) that can be chosen such that we remove (i,j,k). There are $\binom{n}{3}$ ways to choose 3 elements, thus we have $2\binom{n}{3}$ for this case.
For the final case, we have to remove 4 elements, and all 4 must be involved in an inversion (i.e. two 2-cycles). We consider elements labeled $1 < 2 < 3 < 4$. Now we look at the possible sets of two-cycle permutations. We could have: 2143, 3412, or 4321. Now we look at the inversions between the dijoint 2-cycles (since these are the only ones that will matter). For 2143 is not legitimate, since choosing one element from the first and one element from the second cycle would not give an inversion. For 3412 we have two, since if we choose 3 we could choose only 2 (since 4 is not an inversion, and those are the two elements in the the cycle), and if we choose 4 we could only choose 1 (and 1 only 4, and 2 only 3). For 4321 we have four, since for 4 we can choose 3 or 2, and for 1 we can choose 3 or 2 as well. Thus we have a total of 6 pairs that would map to the same removal, thus $6\binom{n}{4}$. Thus, if a_n is the total

number of inversions in involutions of length n, and b_n is the number of involutions of length n, we have

$$a_n = \binom{n}{2} b_{n-2} + 2\binom{n}{3} b_{n-3} + 6\binom{n}{4} b_{n-4}$$

which we can rewrite as $a_n/n! = b_{n-2}/(2(n-2)!) + b_{n-3}/(3(n-3)!) + b_{n-4}/(4(n-4)!)$ thus if we convert this into an exponential generating function and convolve it with the generating function for involutions (which is $e^{x+x^2/2}$), we get the generating function $(x^2/2 + x^3/3 + x^4/4)e^{x+x^2/2}$
We note that the generating function for involutions is just (where c is a constant that will be canceled out in a minute) $e^{z+z^2/2}$ $c(\frac{N}{e})^{N/2}e^{\sqrt{N}} = f(N)$
Thus the asymptotic approximation for our above formula is $(N^2/2)f(N-2) + (N^3/3)f(N-3) + (N^4/4)f(N-4)$, but since in each term, if we look at the $N^{N/2}$ part that dominates, we subtract half of the -2, -3, or -4, and then add the whole thing back, we see that the f(N-4) term is the largest with N/2 + 2 in the exponent, thus the asymptotic approximation is:

$$\frac{N^4(\frac{N-4}{e})^{(N-4)/2}e^{\sqrt{N-4}}}{4(\frac{N}{e})^{N/2}e^{\sqrt{N}}}$$

We note that $(N-4)^{(N-4)/2}$ $N^{(N-4)/2}/e^2$, thus we can simplify the above expression (very nicely!) to:

$$\frac{N^2}{4} * e^{\sqrt{N-4}-\sqrt{N}}$$

which, since as N tends to infinity as per tilde notation, the e term goes to 1, gives that the average number goes to $N^2/4$.
Worked with Matt T, Tim R.