Homework 5: Exercise 6.6

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Let $\mathcal{F}$ be the class of forests and $\mathcal{G}$ the class of general trees. Additionally, let $\mathcal{F}_s$ be the class of forests with *no singleton* trees, $\mathcal{G}_s$ the class of *non-singleton* general trees.

The generating function for non-singleton trees is $G_s(z) = G(z) - z$. Furthermore, a forest of non-singleton trees is a sequence of none-singleton trees, so that

$$\mathcal{F}_s = \textsc{Seq}(\mathcal{G}_s)$$
$$F_s(z) = \frac{1}{1 - G_s(z)} = \frac{1}{1 - (G(z) - z)}$$

In lecture we derived the generating function for general trees by observing a bijection between general trees with N nodes and binary trees with $N - 1$ internal nodes:

$$G(z) = zT(z) \Rightarrow G(z) = \frac{1 - \sqrt{1 - 4z}}{2}$$

Substituting this into the generating function for $\mathcal{F}_s$ gives:

$$\begin{aligned}
F_s(z) &= \frac{1}{1 - (\frac{1-\sqrt{1-4z}}{2} - z)} \\
&= \frac{2}{1 + 2z + \sqrt{1 - 4z}} \cdot \frac{1 + 2z - \sqrt{1 - 4z}}{1 + 2z - \sqrt{1 - 4z}} \\
&= \frac{1 + 2z - \sqrt{1 - 4z}}{2z(z + 2)} \\
&= \frac{1}{z}\left(\frac{1 + 2z}{2(z + 2)} - \frac{\sqrt{1 - 4z}}{2(z + 2)}\right)
\end{aligned}$$

The denominator of the left summand has a root of $-1/2$ with multiplicity 1; therefore, by the rational function transfer theorem, it is asymptotically proportional to a constant times $(-2)^-n$, which is vanishing.

The asymptotics of the right summand (the asymptotically dominant term) can be computed using the radius-of-convergence transfer theorem, with $f(z) = \frac{1}{2(z+2)}$, $\rho = 1/4$, $\alpha = -1/2$. Furthermore, the radius of convergence of f is 2, which is bigger than ρ. The theorem then gives

$$[z^N]\frac{-\sqrt{1 - 4z}}{2(z + 2)} \sim \frac{4^N}{9\sqrt{\pi N^3}}$$

Plugging this back into the generating function equation for $\mathcal{F}_s$ and shifting indices to account for the division by z gives

$$F_s(z) \sim \frac{4^{N+1}}{9\sqrt{\pi(N+1)^3}} \sim \frac{4^{N+1}}{9\sqrt{\pi N^3}}.$$

Furthermore, we know from lecture that the total number of forests with N nodes and no restrictions is

$$[z^n]F(z) \sim \frac{4^N}{\sqrt{\pi N^3}}.$$

Therefore, the ratio of forests with non-singleton trees is

$$\frac{[z^n]F_s(z)}{[z^n]F(z)} \sim \frac{\frac{4^{N+1}}{9\sqrt{\pi N^3}}}{\frac{4^N}{\sqrt{\pi N^3}}} = \frac{4}{9}.$$