Homework 5: Exercise 7.29

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Let $\mathcal{A}$ be the class of arrangements of N elements, where the size function maps every element to N[1]. We will outline a bijection between elements of $\mathcal{A}$ and elements of the labelled product of sets and sequences, so that

$$\mathcal{A} = \text{SET}\,(\mathcal{Z}) \star \text{SEQ}\,(\mathcal{Z})$$

Let $a \in \mathcal{A}$ be an arrangement of N elements. We can uniquely construct a pair $(s,t) \in \text{SET}\,(\mathcal{Z}) \star \text{SEQ}\,(\mathcal{Z})$, with $|s| + |t| = N$, from a: Pick s to be the elements that are *not* in a, and define t to be a sequence of labelled atoms, the order of labels of which follows the order of labels in a.

Conversely, let $(s,t) \in \text{SET}\,(\mathcal{Z}) \star \text{SEQ}\,(\mathcal{Z})$, with $|s| + |t| = N$, and suppose WLOG $|s| = k$. We can construct an arrangement of N elements by ordering the $n - k$ elements that are *not* in s according to the ordering of the labels in t.

Given the bijection outlined above,

$$\mathcal{A} \cong \text{SET}\,(Z) \star \text{SEQ}\,(Z)$$
$$A(z) = \frac{e^z}{1-z}.$$

The number of arrangements of size N can be derived using the binomial convolution of EGFs:

$$\begin{aligned}
N![z^N]A(z) = N![z^N]\frac{e^z}{1-z} &= N!\sum_{k=0}^{n}\binom{N}{k}\left([z^k]e^z\right)\left([z^{n-k}]\frac{1}{1-z}\right) \\
&= N!\sum_{k=0}^{n}\binom{N}{k}\frac{1}{k!} \\
&= \sum_{k=0}^{n}\frac{N!}{k!(N-k)!}\frac{N!}{k!} \\
&= \sum_{k=0}^{n}\frac{N!}{k!} = \sum_{k=0}^{n}\frac{N!}{(N-k)!}.
\end{aligned}$$

[1] This choice of a size function i consistent with p. 133 to the Analytic Combinatorics book: "An arrangement of size n is an ordered combination of (some) elements of $[1..n]$"

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Combinatorially, this sum can be interpreted as follows: In order to make an arrangement of k elements from N elements, we can first order all N elements in a line ($N!$ ways) and then only pick the first k preserving their order, while disregarding the next $N - k$ elements. With this method, for every choice of arrangement of k elements, there are $(N - k)!$ orderings of the disregarded element. Diving by $(N - k)!$ to correct for this "double-counting", the number of arrangements of k elements from N elements is $N!/(N - k)!$. The total number of arrangements is the sum of this over all possible k, which corresponds to the coefficient extracted above.