Homework 5: Exercise 7.45

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Lemma 1. *Let A_n be the accumulative number of inversions in involutions of size n, and let B_n be the number of involutions of size n. Then*

$$A_n = \binom{n}{2} B_{n-2} + 2\binom{n}{3} B_{n-3} + 6\binom{n}{4} B_{n-4}.$$

Proof. The result follows from studying the following kinds of inversions:

- The first term counts the inversions that belong to a cycle of size 2.
  - There are $\binom{n}{2}$ ways of picking two elements to be inverted, and for any choice of such a pair, there are B_{n-2} ways of forming an involution on the remaining elements.
- The second term counts the inversions, in which one element belongs to a cycle of size 1 and the other term belongs to a cycle of size 2.
  - There are $\binom{n}{3}$ ways of picking three elements $i < j < k$. Furthermore, only j can be a fix point of an involution. (Otherwise, suppose i is a fix point and j, k form a cycle. Then applying the involution once does not change the relative order of i, j or i, k, meaning that they cannot have been an inversion. By symmetry, k also cannot be a fix point.) Therefore, any choice of 3 elements results in 2 possible choices of an inversion (i, j and j, k), and for every such choice there are B_{n-3} ways of forming an involution on the remaining elements.
- The third term counts the inversions, in which the elements are derived from two different cycles of size 2.
  - There are $\binom{n}{4}$ ways of picking four elements $a < b < c < d$, and there are 6 ways of assigning two cycles of size 2 to them:
    - $(a, b)(c, d)$: There is no way of picking an inversion with elements from both cycles, since applying the involution does not change the relative location of the elements in different cycles.
    - $(a, c)(b, d)$: Applying the involution gives the following ordering: c, d, a, b. There are 2 ways of picking an inversion with elements from both cycles: If we pick a, the only element form the other cycle creating an inversion is d, and if we pick c, the only element from the other cycle creating an inversion is b.

* $(a,d)(b,c)$: Applying the involution gives the following ordering: d,c,b,a. There are 4 ways of picking an inversion with elements from both cycles: any of a,b, a,c, b,d, and c,d work.

$\square$

Given that the generating function for involutions is $B(z) = e^{z+z^2/2}$, the above lemma translates to the following generating function equations:

$$A(z) = \left(\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4}\right) e^{z+z^2/2}.$$

Asymptotically, the recursive formula from Lemma 1 can be rewritten as

$$A_n \sim \frac{n^2}{2} f(n-2) + \frac{n^3}{3} f(n-3) + \frac{n^4}{4} f(n-4)$$

where, $f(n)$ the the asymptotic number of involutions of size n, derived in lecture:

$$B_n \sim \frac{1}{\sqrt{2\sqrt{e}}} \left(\frac{n}{e}\right)^{n/2} e^{\sqrt{n}} = f(n).$$

Note that in the asymptotic expression for A_n, the $n^4/4 f(n-4)$ dominates:

$$\begin{aligned}
\frac{n^2}{2} f(n-2) &\sim \frac{n^2}{2} \frac{1}{\sqrt{2\sqrt{e}}} \left(\frac{n-2}{e}\right)^{(n-2)/2} e^{\sqrt{n-2}} = O(n^{\frac{n}{2}+1}) \\
\frac{n^3}{3} f(n-3) &\sim \frac{n^3}{3} \frac{1}{\sqrt{2\sqrt{e}}} \left(\frac{n-3}{e}\right)^{(n-3)/2} e^{\sqrt{n-3}} = O(n^{\frac{n}{2}+\frac{3}{2}}) \\
\frac{n^3}{3} f(n-3) &\sim \frac{n^4}{4} \frac{1}{\sqrt{2\sqrt{e}}} \left(\frac{n-4}{e}\right)^{(n-4)/2} e^{\sqrt{n-4}} = O(n^{\frac{n}{2}+2}).
\end{aligned}$$

Therefore, the asymptotic average number of inversions in involutions of size n, given by $A(n)/B(n)$, simplifies to

$$\begin{aligned}
\frac{A_n}{B_n} \sim \frac{n^4}{4} \frac{f(n-4)}{f(n)} &= \frac{n^4}{4} \frac{\left(\frac{n-4}{e}\right)^{(n-4)/2}}{\left(\frac{n}{e}\right)^{n/2}} e^{\sqrt{n-4}-\sqrt{n}} \\
&\sim \frac{n^4}{4} \frac{\left(\frac{n-4}{e}\right)^{(n-4)/2}}{\left(\frac{n}{e}\right)^{n/2}} \\
&\sim \frac{n^4}{4} \frac{n^{(n-4)/2}}{e^2 \cdot e^{(n-4)/2}} \frac{e^{n/2}}{n^{n/2}} \\
&= \frac{n^2}{4}.
\end{aligned}$$