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COS 488/MAT 474
Problem Set 5, Q1

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AofA Exercise 6.6 What proportion of the forests with N nodes have no trees consisting of a single node?

Solution. Let $\mathcal{F}^*$ be the set of forest with no trees consisting of a single node, $\mathcal{G}^*$ the set of trees with ≥ 2 nodes, and $\mathcal{F}_{\geq 1}$ the set of nonempty forests (forests consisting of at least 1 tree). Symbolically, we can write:

$$\mathcal{F}^* = SEQ(\mathcal{G}^*), \qquad \mathcal{G}^* = Z \times \mathcal{F}_{\geq 1}.$$

Now we translate this to generating functions. We have $F_{\geq 0}(z) = F(z) - 1$, because there is only one empty forest. Therefore,

$$G^*(z) = z(F(z) - 1) = zF(z) - z, \qquad F^*(z) = \frac{1}{1 + z - zF(z)}.$$

It is known that $F(z) = \frac{1-\sqrt{1-4z}}{2z}$. Therefore, the generating function for $\mathcal{F}^*$ simplifies to

$$\begin{aligned} F^*(z) &= \frac{1}{1 + z - \frac{1-\sqrt{1-4z}}{2}} \\ &= \frac{2 + 4z - 2\sqrt{1+4z}}{8z + 4z^2} \\ &= \frac{1}{4z} + \frac{3}{4(2+z)} + \frac{-\frac{1}{4z+2z^2}}{(1-4z)^{-1/2}}. \end{aligned}$$

To extract approximations of the coefficients, we can safely disregard the first few terms. For the last term, we use the Corollary to Theorem 5.5, with $\alpha = -\frac{1}{2}, \rho = \frac{1}{4}$, and $f(\rho) = \frac{-1}{4\rho + 2\rho^2} = -\frac{8}{9}$. We thus obtain:

$$[z^N]F^*(z) \sim \frac{-\frac{8}{9}}{\Gamma(-1/2)} 4^N N^{-3/2} = \frac{4^{N+1}}{9N\sqrt{\pi N}}.$$

We already know that

$$[z^N]F(z) \sim \frac{4^N}{N\sqrt{\pi N}}.$$

Therefore, the proportion of forests with N nodes with no trees consisting of a single node is asymptotic to

$$\frac{[z^N]F^*(z)}{[z^N]F(z)} \sim \frac{4}{9}.$$