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COS 488/MAT 474
Problem Set 5, Q2

AofA Exercise 6.42 Internal nodes in binary trees fall into one of three classes: they have either two, one, or zero external children. What fraction of the nodes are of each type, in a random binary Catalan tree of N nodes?

Solution. A leaf is an . By the Corollary on p. 310, the average number of leaves in a binary Catalan tree with N nodes is:

$$\frac{N(N+1)}{2(2N-1)} \sim \frac{N}{4}.$$

Each external node has exactly one parent node, which is either an internal node with one external child, or an internal node with two external children (i.e. a leaf). Therefore, we have

$$(\#\text{ external nodes}) = (\#\text{ internal nodes with 1 external child}) + 2(\#\text{ leaves}).$$

By the Lemma on p. 260, the number of extenal nodes is exactly $N+1$. Therefore, the average number of internal nodes with exactly one external child is equal to:

$$N + 1 - \frac{N(N+1)}{2(2N-1)} = \frac{N^2-1}{2N-1} \sim \frac{N}{2}.$$

The total number of internal nodes add up to N. Therefore, the average number of internal nodes with no external children is equal to:

$$N - \frac{N^2-1}{2N-1} - \frac{N(N+1)}{2(2N-1)} = \frac{N^2-3N+2}{2(2N-1)} \sim \frac{N}{4}.$$

would have preferred if you'd re-derived the N/4 result, but this is fine