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COS 488 - Homework 5 - Question 1

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Let G be the class of trees, and let F_0 be the class of forests with trees consisting of a single node, where the size of an element of G or F_0 is the number of nodes in it. Then, an element of F_0 is a sequence of elements of G that are not the tree consisting of a single node, so we have the following OGF equation:

$$F_0(z) = \frac{1}{1 - (G(z) - z)}$$

From the lecture, we have that $G(z) = \frac{1-\sqrt{1-4z}}{2}$, so

$$F_0(z) = \frac{1}{1 - \frac{1-\sqrt{1-4z}}{2} + z} = \frac{2}{1 + 2z + \sqrt{1-4z}} = \frac{2(1 + 2z - \sqrt{1-4z})}{(1+2z)^2 - (1-4z)} = \frac{1}{2z}\left(\frac{1+2z}{2+z} - \frac{\sqrt{1-4z}}{2+z}\right).$$

The first term in the parentheses will be asymptotically dominated by the second, so we may concentrate on the second term. Then, by the radius-of-convergence transfer theorem with $\rho = 1/4$ and $\alpha = \frac{-1}{2}$, we have that

$$[z^n]zF_0(z) \sim \frac{4^N N^{-3/2}}{2(1/4+2)\Gamma(-1/2)} = \frac{4}{9}\frac{4^{N-1}}{\sqrt{\pi N^3}},$$

typo below, you put # of trees

so $[z^n]F_0(z) \sim \frac{4^N}{\sqrt{\pi N^3}}$.

Since there are asymptotically $\frac{4^N}{\sqrt{\pi N^3}}$ binary trees with N nodes, the proportion of such trees that do not have any trees with one node in them is approximately $\frac{4}{9}$.