5/5

COS 488 - Homework 5 - Question 2

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Let a_2, a_1, and a_0 be the proportion of internal nodes in a random Catalan tree with N nodes that have two, one, and zero external children, respectively.

1. Let T be the class of binary trees, and for each $t \in T$, let $|t|$ be the number of internal nodes of T, and let $f(t)$ be the number of internal nodes of T that have two external children. Let $T(z)$ be the OGF for the number of binary trees with N nodes, and let $\Xi(z)$ be the cumulative cost OGF for f, so

   $$\Xi(z) = \sum_{t \in T} f(t) z^{|t|}.$$

   Then, since there is exactly one tree in which the root node is an internal node with two external children, we have the equation

   $$\begin{aligned} \Xi(z) &= z + \sum_{t_L \in T} \sum_{t_R \in T} (t_L + t_R) z^{|t_L|+|t_R|+1} \\ &= z + z\left(\sum_{t_L \in T} t_L z^{|t_L|}\right)\left(\sum_{t_R \in T} z^{|t_R|}\right) + z\left(\sum_{t_L \in T} z^{|t_L|}\right)\left(\sum_{t_R \in T} t_R z^{|t_R|}\right) \\ &= z + 2z\Xi(z)T(z), \end{aligned}$$

   so $\Xi(z) = \frac{z}{1-2zT(z)}$. Since $T(z) = \frac{1-\sqrt{1-4z}}{2z}$, we have $\Xi(z) = \frac{z}{\sqrt{1-4z}}$.
   Now, by the radius-of-convergence transfer theorem with $\rho = 1/4$ and $\alpha = 1/2$,

   $$[z^n]\Xi(z) \sim \frac{1}{4} \frac{4^N}{\sqrt{\pi N}},$$

   so since there are $\sim \frac{4^N}{N\sqrt{\pi N}}$ total trees with N internal nodes, the average number of internal nodes with two external children in a tree with N internal nodes is $\frac{N}{4}$. Therefore, $a_2 \sim \frac{1}{4}$.

2. In a tree with N internal nodes, there are $N + 1$ external nodes, so we have $2a_2 + a_1 \sim 1$. Since $a_2 \sim \frac{1}{4}$, $a_1 \sim \frac{1}{2}$.

3. Since every internal nodes must have two, one, or zero external children, $a_2 + a_1 + a_0 \sim 1$. Since $a_2 \sim \frac{1}{4}$ and $a_1 \sim \frac{1}{2}$, $a_0 \sim \frac{1}{4}$.