5/5

COS 488 - Homework 5 - Question 4

Matt Tyler

Let A_N be the set of all pairs of involutions σ of length N and pairs (i,j) where $1 \le i < j \le N$ and $\sigma(i) > \sigma(j)$, so that $|A_N|$ is the total number of inversions in all involutions of length N. Let $A'_N \subset A_N$ be the set of all pairs $(\sigma,(i,j))$ where i and j are in the same cycle, let $A''_N \subset A_N$ be the set of all pairs $(\sigma,(i,j))$ where either i or j is fixed by σ (though of course not both), and let $A'''_N \subset A_N$ be the set of all pairs $(\sigma,(i,j))$ where i and j are both in disjoint cycles under σ. Let B_N be the set of all involutions of length N.
Let $\phi_1 : A'_N \to B_{N-2}$ be the map that sends $(\sigma,(i,j))$ to the involution formed by deleting the cycle containing i and j. Then, given any $\tau \in B_{N-2}$, there are exactly $\binom{N}{2}$ elements of A'_N that map to τ under ϕ_1. In particular, given any $1 \le i < j \le N$, there is a unique involution σ such that $\phi_1(\sigma,(i,j)) = \tau$. Therefore, $|A'_N| = \binom{N}{2}|B_{N-2}|$.
Similarly, let $\phi_2 : A''_N \to B_{N-3}$ be the map that sends $(\sigma,(i,j))$ to the involution formed by deleting the cycles containing i and j (one of which has size 1). Then, given any $\tau \in B_{N-3}$, there are exactly $2\binom{N}{3}$ elements of A''_N that map to τ under ϕ_2. In particular, given any $1 \le i < j < k \le N$, there are unique involutions σ_1 and σ_2 such that $\phi_2(\sigma_1,(i,j)) = \phi_2(\sigma_2,(j,k)) = \tau$. Therefore, $|A''_N| = 2\binom{N}{3}|B_{N-3}|$.
Finally, let $\phi_3 : A'''_N \to B_{N-4}$ be the map that sends $(\sigma,(i,j))$ to the involution formed by deleting the two disjoint cycles containing i and j. Then, given any $\tau \in B_{N-4}$, there are exactly $6\binom{N}{4}$ elements of A'''_N that map to τ under ϕ_3. In particular, given any $1 \le i < j < k < l \le N$, there is a unique involution σ that maps i, j, k, and l two disjoint 2-cycles for each of the following six conditions:

1. $\sigma(i) = k$ and $\phi_3(\sigma,(i,l)) = \tau$
2. $\sigma(i) = k$ and $\phi_3(\sigma,(j,k)) = \tau$
3. $\sigma(i) = l$ and $\phi_3(\sigma,(i,j)) = \tau$
4. $\sigma(i) = l$ and $\phi_3(\sigma,(i,k)) = \tau$
5. $\sigma(i) = l$ and $\phi_3(\sigma,(j,l)) = \tau$
6. $\sigma(i) = l$ and $\phi_3(\sigma,(k,l)) = \tau$

(and these are the only six elements of A'''_N that map to τ and have i, j, k, and l be in two disjoint 2-cycles). Therefore, $|A'''_N| = 6\binom{N}{4}|B_{N-4}|$.
Since A_N is the disjoint union of A'_N, A''_N, and A'''_N, we have that

$$|A_N| = \binom{N}{2}|B_{N-2}| + 2\binom{N}{3}|B_{N-3}| + 6\binom{N}{4}|B_{N-4}|,$$

or equivalently

$$\frac{|A_N|}{N!} = \frac{|B_{N-2}|}{2(N-2)!} + \frac{|B_{N-3}|}{3(N-3)!} + \frac{|B_{N-4}|}{4(N-4)!}.$$

Therefore, if $A(z)$ is the CGF for the total number of inversions in all involutions of length N and $B(z)$ is the EGF for the number of involutions of length N, then we have the equation

$$A(z) = \left(\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4}\right)B(z) = \left(\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4}\right)e^{z+\frac{z^2}{2}}.$$

Now, let $a_N = |A_N|$ be the total number of inversions in all involutions of length N, and let $b_N = |B_N|$ be the number of involutions of length N, so that by the CGF equation or by the recurrence found above, the average number of inversions in an involution of length N is

$$\frac{a_N}{b_N} = \frac{\binom{N}{2}b_{N-2} + 2\binom{N}{3}b_{N-3} + 6\binom{N}{4}b_{n-4}}{b_N} \sim \frac{\frac{N^2}{2}f(N-2) + \frac{N^3}{3}f(N-3) + \frac{N^4}{4}f(N-4)}{f(N)}$$

where

$$f(N) = \frac{1}{\sqrt{2\sqrt{e}}}\left(\frac{N}{e}\right)^{\frac{N}{2}} e^{\sqrt{N}}.$$

Then, since the third term in the above sum dominates, we have that

$$\frac{a_N}{b_N} \sim \frac{\frac{N^4}{4}\left(\frac{N-4}{e}\right)^{\frac{N-4}{2}} e^{\sqrt{N-4}}}{\left(\frac{N}{e}\right)^{\frac{N}{2}} e^{\sqrt{N}}} \sim \frac{\frac{N^4}{4}N^{\frac{N-4}{2}}e^{-2}e^{\frac{-N+4}{2}}e^{\sqrt{N-4}}}{N^{\frac{N}{2}}e^{-\frac{N}{2}}e^{\sqrt{N}}} = \frac{N^2}{4}\left(e^{\sqrt{N-4}-\sqrt{N}}\right) \sim \frac{N^2}{4}.$$