4/5, this is better done w/ the transfer thm once you get the GF. your way is nice but gets handwavy toward the end

COS 488 Problem Set #5 Question #1

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Let $T(z) = \frac{1-\sqrt{1-4z}}{2}$ denote the number of trees with N internal nodes, and $F(z)$ be the number of forests with no trees consisting of a single node. Then

$$\begin{aligned}\mathcal{F} &= SEQ(\mathcal{T}(\mathcal{Z}) - \mathcal{Z})\\ F(z) &= \frac{1}{1-\left(\frac{1-\sqrt{1-4z}}{2} - z\right)}\\ &= \frac{2}{1+2z+\sqrt{1-4z}}\\ &= \frac{2(1+2z-\sqrt{1-4z})}{(1+2z)^2-(1-4z)}\\ &= \frac{1+2z-\sqrt{1-4z}}{4z(1+z/2)}\\ &= \frac{1}{2(1+z/2)}\left(1+\frac{1-\sqrt{1-4z}}{2z}\right)\end{aligned}$$

Let $t_n = \frac{1}{n+1}\binom{2n}{n}$ so that $\frac{1-\sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} t_n z^n$ and let $F(z) = \sum_{n=0}^{\infty} f_n z^n$. Then we have that $f_n = \frac{1}{2}\left(-\frac{1}{2}\right)^n + \frac{1}{2}\sum_{k=0}^{n} t_k \left(-\frac{1}{2}\right)^{n-k}$. This in particular gives the recurrence $f_n = \frac{1}{2}(t_n - f_{n-1})$. Let $f'_n = \frac{f_n}{t_n}$ be the proportion of forests of size n that do not contain any trees of size 1. Then we have the recurrence $t_n f'_n = \frac{1}{2}(t_n - t_{n-1}f'_{n-1})$. Note that $t_n n(n+1) = t_{n-1}(2n-1)(2n)$ so that $t_{n-1} = \frac{n+1}{2(2n-1)}t_n$. As a result,

$$2(2n-1)f'_n = 2n-1-\frac{n+1}{2}f'_{n-1}$$

In order to find the asymptote, it suffices to find when this recurrence stabilizes. That is, let $f' = f'_n = f'_{n-1}$. Then we wish to solve for f' as $n \to \infty$.

-1pt, why must it stabilize?

$$\begin{aligned}(4(2n-1)+(n+1))f' &= 2(2n-1)\\ f' &= \frac{4n-2}{9n-3}\end{aligned}$$

As a result, $f' \to \frac{4}{9}$. Hence, as $n \to \infty$, the proportion of forests not containing a tree of size 1 is $4/9$.