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COS 488 Problem Set #5 Question #2

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As noted in the text, a random binary Catalan tree with N internal nodes has on average $\frac{N(N+1)}{2(2N-1)} \sim \frac{N}{4}$ vertices. Each of these has 2 external children, and we know that any binary tree has $N+1$ external nodes, so if N_i is the number of nodes with i external children, we have

$$\begin{aligned} 0 \cdot N_0 + 1 \cdot N_1 + 2 \cdot N_2 &= N + 1 \\ N_1 &= N + 1 - \frac{N(N+1)}{2N-1} \\ &= (N+1)\left(1 - \frac{N}{2N-1}\right) \\ &= \frac{(N+1)(N-1)}{2N-1} \\ &= \frac{N^2-1}{2N-1} \sim \frac{N}{2} \end{aligned}$$

Then, of course, we have that these sum to all the internal nodes of the tree, so

$$\begin{aligned} N_0 + N_1 + N_2 &= N \\ N_0 &= N - \frac{N(N+1)}{2(2N-1)} - \frac{N^2-1}{2N-1} \\ &= \frac{2N(2N-1) - (N^2+N) - (2N^2-2)}{2(2N-1)} \\ &= \frac{N^2-3N+2}{2(2N-1)} \\ &= \frac{(N-1)(N-2)}{2(2N-1)} \sim \frac{N}{4} \end{aligned}$$

would have preferred if you'd re-derived the initial N/4 result, but this is fine