5/5

COS 488 Problem Set #5 Question #4

Tim Ratigan

March 9, 2017

Let a_n be the total number of inversions on the set of involutions of length n, and let b_n the the total number of involutions of length n. Let σ be an involution and let $a < b$ while $\sigma(a) > \sigma(b)$ be an inversion. Then consider the map $\sigma \to \sigma'$ where we delete the cycles containing a and b in σ and restrict the involution to the remaining letters. There are 3 cases:

- a, b are in the same 2-cycle or they are both fixed.

  First of all, a and b cannot be fixed because then they wouldn't be inverted, so we only need to consider the first case.

  In this case, first note that appending the 2-cycle $((n-1)n)$ to the end of $\sigma' \in \mathcal{P}_{n-2}$ gives an inversion on an involution on n letters σ'', and deleting the inversion on this σ'' gives back σ', so this map surjects onto involutions of length $n-2$. We wish to determine the number of elements in the pre-image so that we can count the number of inversions of this form. Given a σ', all the elements in the pre-image of σ' can be obtained by choosing $1 \le j < k \le n$ and inserting the 2-cycle (jk) into σ' as so (with the example $\sigma' = (24)$ and $j = 2, k = 5$)

  $$\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{matrix} \to \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 6 & 4 & 2 & 3 \end{matrix}$$

  Here, everything after j was incremented once and everything after k was incremented twice. One can see that deleting $2, 5$ from the second permutation gets the first one by restricting to the remaining letters.

  As a result, each σ' has $\binom{n}{2}$ elements in the pre-image in this case. This implies that the number of inversions a, b on involutions on n letters where where a and b are in the same cycle is exactly $\binom{n}{2} b_{n-2}$.

- One of a, b is in a 2-cycle, the other is fixed.

  WLOG let a be in a 2-cycle with a' and let b be fixed. Since $a < b$ we must have $a' > b$, so σ sends $(a, b, a') \to (a', b, a)$. Consider any involution σ' on $n-3$ letters. In order to find the pre-images of this under this map, we can insert 3 letters $1 \le i < j < k \le n$ as before into σ' with the attached permutation (ik). Then each of these pre-images maps twice to σ' since both the inversions $i < j \to k > j$ and $j < k \to j > i$ map to σ' when they are deleted.

  As a result, each σ' has $2\binom{n}{3}$ elements in the pre-image in this case. This implies that the number of inversions $a < b \to \sigma(a) > \sigma(b)$ on involutions on n letters where exactly one of a, b are in a 2-cycle is exactly $2\binom{n}{3} b_{n-3}$.

- a, b are in two distinct 2-cycles

  We follow the same procedure as in the previous two cases. Consider an involution $\sigma' \in \mathcal{P}_{n-4}$. We can find the pre-image of this involution by inserting 4 elements $1 \le i < j < k < \ell \le n$ to which we apply 2 disjoint 2-cycles. Then the new permutation restricted to i, j, k, ℓ is either $(ij)(k\ell), (ik)(j\ell)$, or $(i\ell)(jk)$. In the first case, there are no inversions between two elements in disjoint 2-cycles. In the second case, we have $j < k \to \ell > i$ and $i < \ell \to j > k$, so there are 2 such inversions. In the last case, we have $i < j \to \ell > k$, $i < k \to \ell > j$, $j < \ell \to k > i$, and $k < \ell \to j > i$, so there are 4 inversions of this type.

  In total, for each choice of $(i, j, k\ell)$, there are 6 permutations that map back to σ', so the pre-image of σ' has size $6\binom{n}{4}$. As a result, the number of inversions $a < b \to \sigma(a) > \sigma(b)$ on involutions on n letters where a, b are members of disjoint 2-cycles is $6\binom{n}{4} b_{n-4}$.

Hence $a_n = \binom{n}{2}b_{n-2} + 2\binom{n}{3}b_{n-3} + 6\binom{n}{4}b_{n-4}$.

$$\begin{aligned} a_n &= \binom{n}{2}b_{n-2} + 2\binom{n}{3}b_{n-3} + 6\binom{n}{4}b_{n-4} \\ \frac{a_n}{n!} &= \frac{b_{n-2}}{2(n-2)!} + \frac{b_{n-3}}{3(n-3)!} + \frac{b_{n-4}}{4(n-4)!} \end{aligned}$$

If $A(z) = [z^n]\frac{a_n}{n!}$ is the PGF for inversions of involutions and $B(z) = [z^n]\frac{b_n}{n!} = e^{z+z^2/2}$ is the PGF for inversions, then this implies

$$\begin{aligned} A(z) &= \frac{z^2}{2}B(z) + \frac{z^3}{3}B(z) + \frac{z^4}{4}B(z) \\ &= (z^2/2 + z^3/3 + z^4/4)e^{z+z^2/2} \end{aligned}$$

We wish to compute $\frac{a_n}{b_n}$ as $n \to \infty$. We will use the approximation given in the slides: $b_n \sim \frac{1}{\sqrt{2\sqrt{e}}}(n/e)^{n/2}e^{\sqrt{n}}$. Note then that $a_n \sim \frac{1}{\sqrt{2\sqrt{e}}}\left(\frac{n^2}{2}((n-2)/e)^{(n-2)/2}e^{\sqrt{n-2}} + \frac{n^3}{3}((n-3)/e)^{(n-3)/2}e^{\sqrt{n-3}} + \frac{n^4}{4}((n-4)/e)^{(n-4)/2}e^{\sqrt{n-4}}\right)$. It is easy to see that the last term is $O(n^{n/2+4}e^{-(n-4)/2+\sqrt{n-4}})$ which dominates the other two, so

$$\frac{a_n}{b_n} \sim \frac{\frac{n^4}{4}((n-4)/e)^{(n-4)/2}e^{\sqrt{n-4}}}{(n/e)^{n/2}e^{\sqrt{n}}}$$

why does it dominate?

Note $(n-4)^{(n-4)/2} \sim n^{(n-4)/2}(1-4/n)^{(n-4)/2} \sim n^{(n-4)/2}e^{-2} = n^{n/2-2}e^{-2}$. Hence,

$$\begin{aligned} \frac{a_n}{b_n} &\sim \frac{n^{n/2+2}e^{-n/2}e^{\sqrt{n-4}}}{4(n/e)^{n/2}e^{\sqrt{n}}} \\ &= \frac{1}{4}n^2e^{\sqrt{n-4}-\sqrt{n}} \sim \frac{n^2}{4} \end{aligned}$$