David Luo
Exercise 8.57

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Solve the recurrence for p_N given in the proof of Theorem 8.9, to within the oscillating term.

$$p_N = \frac{1}{2^N}\sum_k \binom{N}{k} p_k \ \textit{for}\ N > 1\ \textit{with}\ p_0 = 0,\ p_1 = 1$$

First make p_N valid for all values of N:

$$p_N = \tfrac{1}{2}\delta_{n1} + \frac{1}{2^N}\sum_k \binom{N}{k} p_k$$

$$P(z) = \tfrac{1}{2}z + \sum_{N\geq 0} p_N \frac{z^N}{N!} = \tfrac{1}{2}z + \sum_k \frac{p_k(\frac{z}{2})^N}{k!(N-k)!}$$

$$= \frac{z}{2} + e^{z/2}P(z/2)$$

Then iterate and expand:

$$P(z) = \frac{z}{2} + \frac{z}{4}e^{z/2} + \frac{z}{8}e^{3z/4} + e^{7z/8}P(z/8)$$

$$= z\sum_{k\geq 0} 2^{-(k+1)} e^{(1-2^{-k})z}$$

$$p_N = N![z^N]P(z) = N\sum_{k\geq 0}\frac{1}{2^{k+1}}(1-2^{-k})^{N-1}$$

$$\sim N\sum_{k\geq 0}\frac{1}{2^{k+1}}e^{-N/2^k}$$

And now to isolate the periodic terms:

$$p_N/N \sim \sum_{k\geq 0}\frac{1}{2^{k+1}}e^{-N/2^k}$$

$$= \sum_{k<[lgN]}\frac{1}{2^{k+1}}e^{-N/2^k} + \sum_{k\geq[lgN]}\frac{1}{2^{k+1}}e^{-N/2^k} + O(e^{-N})$$

$$= \sum_{k<0}\frac{1}{2^{k+1+[lgN]}}e^{-N/2^{k+[lgN]}} + \sum_{k\geq 0}\frac{1}{2^{k+1+[lgN]}}e^{-N/2^{k+[lgN]}} + O(e^{-N})$$

$$= \sum_{k<0}\frac{1}{N2^{k+1}}e^{-N/N2^{k-\{lgN\}}} + \sum_{k\geq 0}\frac{1}{N2^{k+1}}e^{-N/N2^{k-\{lgN\}}} + O(e^{-N})$$

$$= \frac{1}{N}\sum_{k<0}\frac{1}{2^{k+1}}e^{-2^{\{lgN\}-k}} + \frac{1}{N}\sum_{k\geq 0}\frac{1}{2^{k+1}}e^{-2^{\{lgN\}-k}} + O(e^{-N})$$

$$p_N = \sum_{k<0} \frac{1}{2^{k+1}} e^{-2^{\{lgN\}-k}} + \sum_{k\geq 0} \frac{1}{2^{k+1}} e^{-2^{\{lgN\}-k}} + O(e^{-N})$$