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Exercise 9.58

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"Describe the graph structure of *partial* mappings, where the image of a point may be undefined. Set up the corresponding EGF equations and check that the number of partial mappings of size N is $(N+1)^N$."

Construct the option for a node to point nowhere as another version of singleton cycles.

Trees: $C = Z * SET(C)$
Partial mappings: $SET(CYC(C)+CYC_1(C))$ since we have one more way for each node to not point to other nodes, or an alternate singleton cycle… $P = SET(CYC(C)+C)$

EGF equations: $C(z) = ze^{C(z)}$, $P(z) = \frac{e^{C(z)}}{1-C(z)}$

Extract coefficients by Lagrange-Bürmann with $f(u) = \frac{u}{e^u}$ and $H(u) = \frac{e^u}{1-u}$:

$$[z^N]P(z) = \tfrac{1}{N}[u^{N-1}]H'(u)(\tfrac{u}{f(u)})^N$$

$$= \tfrac{1}{N}[u^{N-1}]\tfrac{e^{u(N+1)}(2-u)}{(1-u)^2} = \tfrac{1}{N}[u^{N-1}](\tfrac{2e^{u(N+1)}}{(1-u)^2} - \tfrac{ue^{u(N+1)}}{(1-u)^2})$$

$$= 2\sum_k (N+1-k)\tfrac{(N+1)^{k-1}}{k!} - \sum_k (N-k)\tfrac{N^{k-1}}{k!}$$

$$\sim \sum_k (N+1-k)\tfrac{(N+1)^{k-1}}{k!}$$

$$= \sum_{k=0}^{N} \tfrac{(N+1)^k}{k!} - \sum_{k=1}^{N} \tfrac{(N+1)^{k-1}}{(k-1)!}$$

$$= (N+1)^N / N!$$

The answer makes sense. Every entry can be anything or undefined, which means $N+1$ possibilities for N entries, or $(N+1)^N$ total partial mappings.