Analytic Combinatorics Homework 6 Problem 3

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Let $P(z)$ be the EGF for p_z, i.e. $P(z) = \sum_{N \geq 0} p_n \frac{z^n}{N!}$. The formula for the binomial sum for an EGF would give us

$$P(z) = e^{z/2} P\left(\frac{z}{2}\right)$$

if not for the fact that the formula for p_N fails to hold for $N = 0$ and $N = 1$. The formula does give $p_0 = 0$ but if $p_1 = 1$ then the formula gives $p_1 = \frac{1}{2}(p_0 + p_1) = \frac{1}{2}$, so we must add $\frac{z}{2}$ for our generating function to be accurate. Thus, we have

$$P(z) = \frac{z}{2} + e^{z/2} P\left(\frac{z}{2}\right) = \frac{z}{2} + e^{z/2}\frac{z}{4} + e^{3z/4} P\left(\frac{z}{4}\right) = \frac{z}{2} + e^{z/2}\frac{z}{4} + e^{3z/4}\frac{z}{8} + e^{7z/8} P\left(\frac{z}{8}\right)$$
$$= \cdots = \sum_{j=0}^{\infty} e^{z\left(1 - \frac{1}{2^j}\right)} \frac{z}{2^{j+1}}.$$

Now, note that for any c we have

$$ze^{cz} = \sum_{N \geq 0} \frac{c^N z^{N+1}}{N!} = \sum_{N \geq 1} \frac{c^{N-1} z^N}{(N-1)!} = \sum_{N \geq 1} N c^{N-1} \cdot \frac{z^N}{N!}. = \sum_{N \geq 0} N c^{N-1} \cdot \frac{z^N}{N!}.$$

Thus, for any given j, we have

$$e^{z\left(1 - \frac{1}{2^j}\right)} \frac{z}{2^{j+1}} = \frac{1}{2^{j+1}} \sum_{N \geq 0} N \left(1 - \frac{1}{2^j}\right)^{N-1} \cdot \frac{z^N}{N!}.$$

Thus, we have

$$P_N = \sum_{j \geq 0} \frac{N}{2^{j+1}} \left(1 - \frac{1}{2^j}\right)^{N-1} \sim N \sum_{j \geq 0} \frac{e^{-N/2^j}}{2^{j+1}},$$

where we use the exp-log trick in the last step (together with the Taylor series for log), as in Slide 59. Thus, we have

$$P_N \sim N \sum_{j \geq 0} \frac{e^{-N/2^j}}{2^{j+1}} = N \sum_{j \geq -\lfloor \lg N \rfloor} \frac{e^{-N/2^{j+\lfloor \lg N \rfloor}}}{2^{j+1+\lfloor \lg N \rfloor}} = N \sum_{j \geq -\lfloor \lg N \rfloor} \frac{e^{-2^{\{\lg N\}-j}}}{N \cdot 2^{j+1-\{\lg N\}}}$$
$$= \sum_{j \geq -\lfloor \lg N \rfloor} \frac{e^{-2^{\{\lg N\}-j}}}{2^{j+1-\{\lg N\}}}.$$