Analytic Combinatorics Homework 6 Problem 4

Eric Neyman
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Observe that a partial mapping can be thought of as a digraph much like that of a mapping, except that some vertices might not have arrows coming out of them. We can think of these as roots of Cayley trees, and their connected components as Cayley trees. Thus, a partial mapping is a set of cycles of Cayley trees (i.e. the components without any unmapped nodes), together with a set of Cayley trees. Let L denote the class of partial mappings. We have

$$L = SET(CYC(C)) \star SET(C).$$

Now, $SET(CYC(C))$ has the generating function $\frac{1}{1-C(z)}$ (see Slide 55), while $SET(C)$ has the generating function $e^{C(z)}$. Thus, we have $L(z) = \frac{e^z}{1-C(z)}$. Applying the Lagrange inversion formula with $f(u) = \frac{u}{e^u}$ as usual and $H(u) = \frac{e^u}{1-u}$, we have

$$[z^N]L(z) = \frac{1}{N}[u^{N-1}]\frac{(2-u)e^u}{(1-u)^2}e^{uN}.$$

Now, $(1-u)^2$ has generating function $\sum_{n\geq 0} n \cdot u^n$, so multiplying by $2-u$ gives a generating function whose n-th coefficient is $2n-(n-1) = n+1$, i.e. $\sum_{n\geq 0}(n+1)u^n$. Convolving with $e^u \cdot e^{uN} = e^{u(N+1)}$ (whose generating function is $\sum_{n\geq 0} \frac{(N+1)^n}{n!}u^n$ and taking the $(N-1)$-th coefficient gives us

$$\begin{aligned}
[z^N]L(z) &= \frac{1}{N}\sum_{k=0}^{N-1}\frac{(N-1-k+2)(N+1)^k}{k!} = \frac{1}{N}\left(\sum_{k=0}^{N-1}\frac{(N+1)\cdot(N+1)^k}{k!} - \sum_{k=0}^{N-1}\frac{k(N+1)^k}{k!}\right) \\
&= \frac{1}{N}\left(\sum_{k=0}^{N-1}\frac{(N+1)^{k+1}}{k!} - \sum_{k=1}^{N-1}\frac{(N+1)^k}{(k-1)!}\right) = \frac{1}{N}\left(\sum_{k=0}^{N-1}\frac{(N+1)^{k+1}}{k!} - \sum_{k=0}^{N-2}\frac{(N+1)^{k+1}}{k!}\right) \\
&= \frac{1}{N}\cdot\frac{(N+1)^N}{(N-1)!} = \frac{(N+1)^N}{N!}.
\end{aligned}$$

Since we were working with an EGF, the number of partial mappings of size N is $N![z^N]L(z) = (N+1)^N$, as desired.